Physikalisches Institut
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Theoretische Physik

Exercise Sheet 9
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SS 17

Exercises General Relativity and Cosmology

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–Homework–

1 Perturbative gravity & Gravitational waves (25 points)

In the lecture it has been discussed the Newtonian limit for GR: For slowly moving particles under a weak and static gravitational field, General relativity reduces to Newtonian gravity. Now we want to describe gravity effects beyond Newtonian theory. Hereby we only consider here a weak gravitational field.

To make a perturbative description of gravity, consider a coordinate system (U, x) of the space-time manifold M, in which the metric g takes the form $g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu$ with

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x), \tag{1}$$

where η is the Minkowski metric and $h_{\mu\nu}(x)$ is a small perturbation, i.e., $|h_{\mu\nu}(x)| \ll 1$ and $|\partial_\rho h_{\mu\nu}(x)| \ll 1$ for all values in $x(U)$ [1]. Hence, it is sufficient to work in linear order in $h_{\mu\nu}(x)$. In particular, we can raise and lower indices of $h_{\mu\nu}(x)$ with the Minkowski metric $\eta_{\mu\nu}$.

a) Consider a change of coordinates $\psi : x(U) \to y(V)$ to a new coordinate system (V, y) given by *(4 points)*

$$\psi : x^\mu \mapsto y^\mu(x) = x^\mu + \epsilon^\mu(x), \tag{2}$$

with $|\epsilon^\mu(x)| \ll 1$ and $|\partial_\nu \epsilon^\mu(x)| \ll 1$ for all values in $x(U)$. Show that the inverse coordinate change ψ^{-1} to relevant leading order reads

$$\psi^{-1} : y^\mu \mapsto x^\mu(y) = y^\mu - \epsilon^\mu(y). \tag{3}$$

Further, express the metric $g = g'_{\mu\nu}(y)\mathrm{d}y^\mu \mathrm{d}y^\nu$ in the coordinate system (V, y) to leading order. We might refer to the first order perturbation of the metric in this new coordinate system as a *gauge transformation* in the linearized theory.

b) Show that the Ricci tensor $\mathrm{Ric} = R_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu$ in the coordinate system (U, x) is given by *(4 points)*

$$R_{\mu\nu}(x) = \frac{1}{2}\left(\partial_\lambda \partial_\mu h^\lambda{}_\nu + \partial_\lambda \partial_\nu h^\lambda{}_\mu - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h\right), \tag{4}$$

where $h = h^\mu{}_\mu$ and $\Box = \eta^{\mu\nu}\partial_\mu \partial_\nu$.

[1] More generally, one can consider perturbations over a general spacetime background, i.e. $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$. For the present discussion we restrict to $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$.

Since we are free to work with whatever coordinates we like, we can use coordinate transformations as in item a) to simplify the problem. For matters of simplification, people like introducing the *trace reversed perturbation* defined by

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}. \tag{5}$$

In particular, it is possible to enforce the *Lorenz gauge*

$$\partial_\mu \bar{h}^{\mu\nu} = 0 \ . \tag{6}$$

c) What coordinate transformation in a) should be performed to achieve the Lorenz gauge condition? *(2 points)*

d) Show that in this gauge we obtain the linearized Einstein equation is *(2 points)*

$$\Box \bar{h}_{\mu\nu} = -16\pi G_N T_{\mu\nu}. \tag{7}$$

e) Verify that the plane wave *(2 points)*

$$h_{\mu\nu}(x) = a_{\mu\nu} \cos(k \cdot x + \phi) \quad \text{with} \quad a \cdot b = \eta(a^\mu \partial_\mu, b^\nu \partial_\nu) = a^\mu b_\mu \tag{8}$$

with constant k^μ, constant symmetric $a_{\mu\nu}$ and constant phase ϕ solves the linearized Einstein equation (7) in the vacuum, i.e. for $T_{\mu\nu} = 0$, if $k^\mu \partial_\mu$ is light-like. Further show that the gauge condition (6) implies

$$k_\lambda a^\lambda{}_\mu = \frac{1}{2} k_\mu a^\lambda{}_\lambda. \tag{9}$$

f) Assume that $a_{\mu\nu}$ fulfills the gauge condition (9). Show that *(1 point)*

$$\tilde{a}_{\mu\nu} = a_{\mu\nu} + k_\mu b_\nu + k_\nu b_\mu \tag{10}$$

with constant b_μ fulfills this condition as well.

g) With the help of the discovered gauge transformations count the physical degrees of freedom of $a_{\mu\nu}$. *(2 points)*

Hint: Note that the transformation of $a_{\mu\nu}$ to $\tilde{a}_{\mu\nu}$ in eq. (10) *arises from a suitable coordinate transformation. This fact you do no need to show.*

Now we want to solve Einstein's equation when $T_{\mu\nu}$ is non-vanshing due to a matter source. Recall from your knowledge on Electrodynamics, an equation like (7) can be solved by using Green's functions, that is

$$\bar{h}_{\mu\nu}(x) = -16\pi G_N \int \mathrm{d}^4 y G(x-y) T_{\mu\nu}(y). \tag{11}$$

h) Argue that the Green function required in (11) leads to *(2 points)*

$$\bar{h}_{\mu\nu}(t, \vec{x}) = 4G_N \int \mathrm{d}^3 y \frac{1}{|\vec{x} - \vec{y}|} T_{\mu\nu}(t^{ret}, \vec{y}), \ \text{with } t^{ret} = t - |\vec{x} - \vec{y}|. \tag{12}$$

i) Consider the observer —with coordinate $x(p)$— to be very far away from the matter source responsible of $T_{\mu\nu}$. Say at a distance r to the observer. Under this assumption, show that (12) can be approximated as (*2 points*)

$$\bar{h}_{\mu\nu}(x) = -\frac{4G_N}{r}\int \mathrm{d}^3y T_{\nu}(t^{ret}, \vec{y}). \tag{13}$$

j) Using the Lorenz gauge condition $\partial_\mu \bar{h}^{\mu\nu} = 0$, derive the *quadrupole formula* for the metric perturbation h in terms of the quadrupole momentum tensor Q^{ij} (*2 points*)

$$\bar{h}^{ij}(x) = -\frac{2G_N}{r}\frac{\mathrm{d}^2 Q^{ij}(t)}{\mathrm{d}t^2}\Big|_{t=t^{ret}}, \text{ with } Q^{ij}(t) = \int \mathrm{d}^3y T^{00}(t,\vec{y})y^i y^j. \tag{14}$$

Finally we examine a special case for production of gravitational waves. Consider two stars of mass $m_\star$ in a circular orbit in the y^1, y^2 plane, at a distance R from their common center of mass. We can write a path for the star a and b as

$$y_a^1(t) = R\cos\Omega t, \quad y_a^2(t) = R\sin\Omega t, \tag{15}$$
$$y_b^1(t) = -R\cos\Omega t, \quad y_b^2(t) = -R\sin\Omega t. \tag{16}$$

The corresponding energy density of the binary system is given by

$$T^{00}(y) = m_\star\delta^3(\vec{y})[\delta(y^1 - R\cos\Omega t)\delta(y^2 - R\sin\Omega t) + \delta(y^1 + R\cos\Omega t)\delta(y^2 + R\sin\Omega t)]. \tag{17}$$

k) Using (14), find the expression for the perturbed metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ —due to a binary system— for an observer standing on earth. (*2 points*)