Exercises on Advanced Topics in String Theory

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Exercise Sheet 1

1.1 Euler number of S^2 and D^2 *(3,5 points)*

In this exercise we want to discuss two methods of calculating the Euler number χ. The Euler number is a topological invariant and therefore the same for spaces which are homeomorphic to each other. In string pertubation theory the Euler number is related to the order of the loop correction of string interactions. One way to calculate χ is the following: Let X be a subset of $\mathbb{R}^3$, which is homeomorphic to a polyhedron K. Then the Euler number $\chi(X)$ of X is defined by

$$\chi(X) = (\#\,\text{of vertices in } K) - (\#\,\text{of edges in } K) + (\#\,\text{of faces in } K) \tag{1}$$

(a) Calculate χ for the 2-sphere S^2 and the disk D^2 embedded in $\mathbb{R}^3$ by finding polyhedrons homeomorphic to S^2 and D^2. *(0,5 points)*

An alternative definition of the Euler number for a Riemann surface Σ is given by the Gauss-Bonnet theorem

$$\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma \sqrt{h}\mathcal{R}\mathrm{d}^2\sigma + \frac{1}{2\pi}\int_{\partial\Sigma} k\mathrm{d}s. \tag{2}$$

$\mathcal{R}$ denotes the Ricci scalar of Σ and is the determinant of the metric $h = \det(h_{\alpha\beta})$ given in terms of the coordinates σ_1, σ_2 on Σ. k is the curvature along the geodesic s on the boundary of the surface $\partial\Sigma$

(b) Find a parametrisation for S^2 and D^2 in $\mathbb{R}^3$ and calculate the metric $g_{\mu\nu}$ and Christoffel symbols $\Gamma^\kappa{}_{\mu\nu} = \frac{1}{2}g^{\kappa\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$ for the manifolds. *(1 point)*

(c) Calculate $\mathcal{R}$ for S^2 and D^2. *(1 point)*
Hint: $\mathcal{R} = g^{\mu\nu}R_{\mu\nu}$ and $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ with $R^\kappa{}_{\lambda\mu\nu} = \partial_\mu\Gamma^\kappa{}_{\nu\lambda} - \partial_\nu\Gamma^\kappa{}_{\mu\lambda} + \Gamma^\eta{}_{\nu\lambda}\Gamma^\kappa{}_{\mu\eta} - \Gamma^\eta{}_{\mu\lambda}\Gamma^\kappa{}_{\nu\eta}$

(d) What is the boundary for S^2 and D^2? Calculate k for the geodesics on S^2 and D^2 and find the Euler numbers using the above results and the Gauss-Bonnet Theorem.
hint: If the geodesic $\vec{r}(s)$ is parametrised by the arc length parameter s the curvature is defined by $k = \|\frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\|$, with $\vec{T}$ the unit Tangentvector on $\vec{r}(s)$. *(1 point)*

1.2 Reduction to moduli of the string partition function *(6,5 points)*

The string partition function for a worldsheet with the topology of a compact Riemann surface with genus h is given by

$$Z = \mathcal{N}\int \mathcal{D}g_{ab}\mathcal{D}X^\mu \exp\left(-\int \mathrm{d}^2\sigma\sqrt{g}(\frac{1}{2}g^{ab}\partial_a X^\mu\partial_b X_\mu + \frac{1}{4\pi}\lambda\mathcal{R})\right), \tag{3}$$

with $\mathcal{N}$ some normalisation, σ^1, σ^2 coordinates on the worldsheet, $X^\mu(\sigma^1, \sigma^2)$ the mapping from the worldsheet to the target space, λ the string coupling, $\mathcal{R}$ the Ricci scalar of the worldsheet, $g = \det g_{ab}$ and g_{ab} a metric on the wordsheet.

(a) Show that the partition function in (3) is equivalent to *(0,5 points)*

$$Z = \mathcal{N} \mathrm{e}^{\lambda(2-2h)} \int \mathcal{D}g_{ab} \mathcal{D}X^\mu \exp\left(- \int \mathrm{d}^2\sigma \sqrt{g} (\frac{1}{2} g^{ab} \partial_a X^\mu \partial_b X_\mu) \right). \tag{4}$$

The integral in (3) is highly divergent because one integrates infinitely many times over conformaly equivalent surfaces. We need to extract the divergent part and only integrate over physically inequivalent metrics. Let $\mathcal{G}_h$ denote the space of admissible metrics on a compact Riemann surface with genus h. Then

$$\langle \delta g_{ab}^{(1)}, \delta g_{ab}^{(2)} \rangle = \int \mathrm{d}^2\sigma \sqrt{g} g^{ac} g^{bd} \delta g_{ab}^{(1)} \delta g_{cd}^{(2)}. \tag{5}$$

defines a scalar $\langle \cdot, \cdot \rangle$ product of two infinitesimal variations $\delta g_{ab}^{(1)}$ and $\delta g_{ab}^{(2)}$ of the metric g_{ab}. More precise $\delta g_{ab}^{(1)}$ and $\delta g_{ab}^{(2)}$ are elements of the tangent space $T_g(\mathcal{G}_h)$ at the the point $g_{ab} \in \mathcal{G}_h$.

(b) Recall that Weyl transformations and diffeomorphisms of the metric do not change the physical results and are therefore symmetries of the worldsheet. Show that under the combined action of

- a Weyl scaling $g_{ab} \to \mathrm{e}^\phi g_{ab}$
- and diffeomorphism generated by a vector field $\vec{v}$

the infinitesimal variation of g_{ab} is given by

$$\delta g_{ab} = \delta\phi g_{ab} + \nabla_a v_b + \nabla_b v_a. \tag{6}$$

Split δg_{ab} into a trace part $\delta g_{ab}^{\mathrm{W}}$ and a symmetric traceless part $\delta g_{ab}^{\mathrm{D}}$ and show that the the norm of δg_{ab} is given by

$$||\delta g||^2 = ||\delta g^{\mathrm{W}}||^2 + ||\delta g^{\mathrm{D}}||^2. \tag{7}$$

How do $\delta g_{ab}^{\mathrm{W}}$ and $\delta g_{ab}^{\mathrm{D}}$ look like explicitly? *(3 points)*
Hint: Define an operator P which maps vectors $\vec{u} \in T_g(\mathcal{G}_h)$ to second-rank symmetric traceless tensors $u_a \to (P\vec{u})_{ab}$.

(c) Conformal killing vectors (CKV) $\vec{v}_{CKV}$ fulfill the condition

$$\nabla_a v_b + \nabla_b v_a = \lambda g_{ab}, \quad \text{with} \quad \lambda \in \mathbb{R}. \tag{8}$$

Show that CKV are elements $Ker(P)$ of and can be replaced by Weyl scalings. *(1 point)*

We are interested in the subspace $\mathcal{M}_h \subset \mathcal{G}_h$, containing all conformal equivalence classes of metrics, called the *moduli space*. Denoting the set of Weyl scalings as $Weyl$ and the set of diffeomorphisms as $Diff$, the moduli space is represented as

$$\mathcal{M}_h \sim \frac{\mathcal{G}_h}{Weyl \times Diff}. \tag{9}$$

In general the infinitesimal variation of a metric $g(t_i) \in \mathcal{G}_h$ is given by

$$\delta g_{ab} = \delta g_{ab}^{\mathrm{W}} + \delta g_{ab}^{\mathrm{D}} + \delta t_i \frac{\partial}{\partial t_i} g_{ab}, \tag{10}$$

where t_i are the moduli parameter.

(d) Split the term $\delta t_i \frac{\partial}{\partial t_i} g_{ab}$ into a trace and traceless symmetric part and define an operator T_{ab}^i which acts on δt_i and maps it to the traceless symmetric term of $\delta t_i \frac{\partial}{\partial t_i} g_{ab}$. *(2 points)*