Physikalisches Institut
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Exercise 2
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# Exercises on Advanced Topics in String Theory

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http://www.th.physik.uni-bonn.de/people/fierro/StringSS18/
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*"Lo verdaderamente nuevo da miedo o maravilla."*

Julio Cortázar, *Historia de Cronopios y Famas*

## 1 Modularity on $T^2$

In string pertubation theory the one loop graph for closed strings has topology $\Sigma_{g=1} \simeq T^2$. A torus can be constructed by modding out a two dimensional lattice $\Lambda$ out of $\mathbb{C}$. This means that points in $\mathbb{C}$ differing by $\lambda \in \Lambda$ are identified

$$\mathbb{C} \ni z \sim z + \lambda\,. \tag{1}$$

For a given lattice $\Lambda = \{\lambda = n_1\ell + n_2\tau\ell | n_1, n_2 \in \mathbb{Z}\}$ with lattice vectors $\ell$ and $\tau\ell$, we can specify the torus by its moduls $\tau \in \mathbb{C}$. However $SL(2,\mathbb{Z})$ transformations on $\tau$ describe the same torus. The group $SL(2,\mathbb{Z})$ is the set of matrices given by

$$SL(2,\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z},\, ad - bc = 1 \right\} \tag{2}$$

They act on $z \in \mathbb{C}$ by $\gamma z = \frac{az+b}{cz+d}$. The generators of $SL(2,\mathbb{Z})$ are given by $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We further define $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm \mathbb{1}\}$ and the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$.

(a) How does $T$ and $S$ act on $\mathbb{H}$? Why is it enough to consider $PSL(2,\mathbb{Z})$? *(2 points)*

(b) Show that there exists a $\gamma_0 \in SL(2,\mathbb{Z})$ such that $\mathrm{Im}(\gamma z) \leq \mathrm{Im}(\gamma_0 z)$ for all $\gamma \in SL(2,\mathbb{Z})$ and fixed $z \in \mathbb{H}$. *(2.5 points)*

(c) Show that $|\gamma_0 z| \geq 1$.*Hint: Apply an $S$ transformation on $\gamma_0 z$* *(1.5 points)*

(d) Show that $|T^n \gamma_0 z| \geq 1$ for any $n \in \mathbb{Z}$ and that one can use $T$ transformations to achieve $-\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}$. What is the fundamental domain $\mathcal{F}$ of $SL(2,\mathbb{Z})$? *(4 points)*

(e) Argue that two moduli $\tau$ and $\tau'$ differing by $SL(2,\mathbb{Z})$ transformations describe the same torus. *(2 points)*

The torus partition function $A_0^{g=1}$ describes the one loop vacuum amplitude of closed strings. It is given by[1]

$$A_0^{g=1} = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{4(\mathrm{Im}(\tau))^2} Z(\tau,\overline{\tau}), \tag{3}$$

where

$$Z(\tau,\overline{\tau}) = \frac{V_{26}}{\ell_s^{26}} \frac{1}{(\mathrm{Im}(\tau))^{12}} |\eta(\tau)|^{-48}, \quad \text{with} \quad \eta(\tau) = \mathrm{e}^{\pi i\tau/12} \prod_{n=1}^{\infty}(1-\mathrm{e}^{2\pi i n\tau}) \tag{4}$$

and $\tau \in \mathbb{C}$ is the moduls of the $T^2$ such that

$$\mathbb{C} \ni z \sim z+1 \quad \text{and} \quad z \sim z+\tau. \tag{5}$$

(f) Show $\mathrm{Im}(\tau)$ is the area of the $T^2$ with moduli $\tau$. How does the measure $\frac{\mathrm{d}^2\tau}{4(\mathrm{Im}(\tau))^2}$ transform under $SL(2,\mathbb{Z})$? *(3 points)*

(g) Show the transformation properties of $\eta(\tau)$ under the action of the generators $S$ and $T$ of $SL(2,\mathbb{Z})$:

$$\eta(\tau+1) = \mathrm{e}^{\pi i/12}\eta(\tau) \quad \text{and} \quad \eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau) \tag{6}$$

and use the result to show that $A_0^{g=1}$ is invariant under $SL(2,\mathbb{Z})$. *(5 points)*

The torus partition function is **modular invariant** due to its transformation properties under the **modular group** $PSL(2,\mathbb{Z})$. Modular invariance of closed string amplitudes can be used to uncover inconsistencies of string theories. For example we will later see, that modular invariance implies spacetime supersymmetry for the superstring.

# 2 Reduction to moduli of the string partition function

This exercise is a continuation of section 1.2 of **Exercise sheet 1**. Let $\Sigma_h$ be a compact oriented Riemann surface of genus $h$. We are interested in the subspace $\mathcal{M}(\Sigma_h) \subset \mathcal{G}(\Sigma_h)$, containing all conformal equivalence classes of metrics, called the **moduli space**. Denoting the set of Weyl scalings as $Weyl(\Sigma_h)$ and the set of diffeomorphisms as $Diff(\Sigma_h)$, the moduli space is represented as

$$\mathcal{M}(\Sigma_h) = \frac{\mathcal{G}(\Sigma_h)}{Weyl(\Sigma_h) \times Diff(\Sigma_h)}. \tag{7}$$

In general the infinitesimal variation of a metric $g(t_i) \in \mathcal{G}_{\Sigma_h}$ is given by

$$\delta g_{ab} = \delta g_{ab}^{\mathrm{W}} + \delta g_{ab}^{\mathrm{D}} + \delta t_i \frac{\partial}{\partial t_i} g_{ab}, \tag{8}$$

where $t_i \in \mathcal{M}(\Sigma_h)$ are the **moduli parameter**. In order to integrate the string partition function only over physically inequivalent metrics, we need to find an appropriate gauge slice in $\mathcal{G}(\Sigma_h)$. Therefore we first need to find a slice $\tilde{\mathcal{G}}(\Sigma_h)$ which contains all equivalence classes of metrics related by Weyl transformations. Then the gauge slice lies in $\tilde{\mathcal{G}}(\Sigma_h)$ and is chosen in such a way that a transformation $\exp(P\vec{v})$ on a point $\tilde{g}_{ab} \in \{\text{gauge slice}\}$ leads to a point $\hat{g}_{ab}$ still in $\tilde{\mathcal{G}}(\Sigma_h)$ but no longer in the gauge slice.

(e) Consider a point $\tilde{g}_{ab} \in \tilde{\mathcal{G}}_(\Sigma_h)$. Is it possible to act on $\tilde{g}_{ab}$ with an element $\in Diff(\Sigma_h)$ in such a way, that on leaves the slice $\tilde{\mathcal{G}}(\Sigma_h)$?Explain why! *(1 point)*

[1]You will derive this result in the next section.

(f) Let us denote an infinitesimal variation changing the conformal equivalence class by $\delta g^{\perp}_{ab}$. It is therefore a tangent vector in the tangent space of $\mathcal{M}(\Sigma_h)$. Why must $\delta g^{\perp}_{ab}$ be traceless? Show that $\delta g^{\perp}_{ab} \in Ker(P^{\dagger})$. *Hint: How is the angle between the tangent vectors $(P\vec{v})_{ab}$ and $\delta g^{\perp}_{ab}$?* *(1.5 points)*

(g) Let $\psi^{\alpha}_{ab}$, $\alpha = 1, ..., \dim Ker(P^{\dagger})$ be an orthonormal basis for $Ker(P^{\dagger})$ and decompose $T^i_{ab}\delta t_i$ into a linear combination of basis vectors of $Ker(P^{\dagger})$ and vectors of $Range(P)$. You should arrive at

$$T^i_{ab}\delta t_i = \langle\psi^{\alpha}, T^i\rangle\psi^{\alpha}_{ab}\delta t_i + \frac{\langle P\vec{v}, T^i\rangle}{||P\vec{v}||^2}(P\vec{v})_{ab}\delta t_i. \tag{9}$$

Show that the norm of $\delta g_{ab}$ is given by

$$||\delta g||^2 = ||\delta\tilde{\phi}||^2 + ||P\tilde{v}||^2 + \langle\psi^{\alpha}, T^i\rangle\langle\psi^{\alpha}, T^j\rangle\delta t_i\delta t_j, \tag{10}$$

with

$$\delta\tilde{\phi} = \delta\phi + \nabla_c v^c + \frac{1}{2}\left(g^{cd}\delta t_i\frac{\partial}{\partial t_i}g_{cd}\right) \quad \text{and} \quad \tilde{v} = \left(1 + \frac{\langle P\vec{v}, T^i\delta t_i\rangle}{P\vec{v}, P\vec{v}}\right)\vec{v}. \tag{11}$$

*(2 points)*

In order to change the path integral variables from $g_{ab}$ to $\phi$, $\vec{v}$ and $t_i$ we use the relation

$$\begin{aligned} 1 &= \int \mathcal{D}g_{ab}\exp(-||\delta g||^2/2) \\ &= J\int \mathcal{D}\phi\mathcal{D}v'^a\mathrm{d}t^1...\mathrm{d}t^n\exp\left(-[||\delta\tilde{\phi}||^2 + ||P\tilde{v}'||^2 + \langle\psi^{\alpha}, T^i\rangle\langle\psi^{\alpha}, T^j\rangle\delta t_i\delta t_j]/2\right) \end{aligned} \tag{12}$$

to calculate the Jacobian $J$. Notice that $\vec{v}'$ denotes elements from $Range(P)$. Since elements from $Ker(P)$ are orthogonal to $\vec{v}'$ we can decompose the volume of the diffeomorphism group $V_{Diff}$ into $V^{\perp}_{Diff} \times V^{CKV}_{Diff}$. Let $\chi_i$, $i = 1, ..., \dim Ker(P)$ be a basis for $Ker(P)$, then one can show that

$$V^{\perp}_{Diff} = V_{Diff}\left(\det\langle\chi_i, \chi_j\rangle\right)^{-1/2}. \tag{13}$$

From (12) one can show that the Jacobian for the path integral should be given by $J = \det^{1/2}(P^{\dagger}P)\frac{\det\langle\psi^i, T^j\rangle}{\det\langle\psi^i, \psi^j\rangle}$.

(h) Show that

$$\int \mathcal{D}g_{ab} \to V_{Diff}\int \mathcal{D}\phi\mathrm{d}t^1...\mathrm{d}t^n\left(\frac{\det(P^{\dagger}P)}{\det\langle\chi^i, \chi^j\rangle}\right)^{1/2}\frac{\det\langle\psi^i, T^j\rangle}{\det\langle\psi^i, \psi^j\rangle}. \tag{14}$$

*(0.5 points)*

(i) Express the number of real moduli $n$ by the genus $h$ for a compact Riemann surface with no crosscaps and $h \geq 2$. *Hint: There are no CKV for compact Riemann surfaces with $h \geq 2$.* *(1 point)*

In the critical dimension ($D = 26$ for the bosonic string) the integrand becomes independent from $\phi$ and the integral $\int \mathcal{D}\phi = V_{Conf}$ can be absorbed into the normalization. It can be shown that the integral over the mappings $X^{\mu}$ is given by

$$\int \mathcal{D}X^{\mu}\exp\left(-\int \mathrm{d}^2\sigma\sqrt{g}(\frac{1}{2}g^{ab}\partial_a X^{\mu}\partial_b X_{\mu}\right) = \mathcal{V}\left(\frac{\int \mathrm{d}^2\sigma\sqrt{g}}{2\pi}\right)^{13}(\det\Delta_g)^{-13}, \tag{15}$$

with $\mathcal{V}$ the volume of space time and $\Delta_g = -\frac{1}{\sqrt{g}}\partial_a\sqrt{g}g^{ab}\partial_b$. Putting the previous results together we find that the partition function in **Exercise sheet 1** can be expressed by

$$A^h_0 = \mathcal{V}\mathrm{e}^{\lambda(2-2h)}\int_{\mathcal{M}_h}\mathrm{d}t^1...\mathrm{d}t^n\left(\frac{\det(P^{\dagger}P)}{\det\langle\chi^i, \chi^j\rangle}\right)^{1/2}\frac{\det\langle\psi^i, T^j\rangle}{\det^{1/2}\langle\psi^i, \psi^j\rangle}\left(\frac{2\pi}{\int \mathrm{d}^2\sigma\sqrt{g}}\det\Delta_g\right)^{-13}. \tag{16}$$

(h) Now that we have the general expression for the partition function of a compact Riemann surface let us apply the results to the $h = 1$ case. The worldsheet hast the topology of a torus and $A_0^h$ is the torus partition function.

(i) Show that $\chi^1 = (1, 0)^{\mathrm{T}}$ and $\chi^2 = (0, 1)^{\mathrm{T}}$ are a possible choice for $Ker(P)$. (*0.5 points*)

(ii) Argue that $n = 2$ and show that $T^i_{ab}$ are given by

$$T^1_{ab} = \begin{pmatrix} -\tau_1 & 1-\tau_1 \\ 1-\tau_1 & \tau_1 \end{pmatrix} \quad \text{and} \quad T^2_{ab} = \begin{pmatrix} -\tau_2 & -\tau_2 \\ -\tau_2 & \tau_2 \end{pmatrix}, \tag{17}$$

where $\tau_1$ and $\tau_2$ are the moduli of the torus with the $g_{ab} = |\mathrm{d}\sigma^1 + (\tau_1 + i\tau_2)\mathrm{d}\sigma^2|^2$. Why do $T^1_{ab}, T^2_{ab}$ form a possible basis for $Ker(P^\dagger)$. *Hint: Since the metric is flat* $(P^\dagger T^i)_b = -2\partial^a T^i_{ab}$. (*1.5 points*)

(iii) Next calculate $\det\langle\chi^i, \chi^j\rangle$ and $\frac{\det\langle\psi^i, T^j\rangle}{\det^{1/2}\langle\psi^i, \psi^j\rangle}$ and show that $\det(P^\dagger P) = (\det(2\Delta_g))^2$. *Hint: First show* $(P^\dagger P)_{ab}v^b = 2\delta_{ab}\Delta_g v^b$ (*1 point*)

(iv) Use $\det(2\Delta_g) = \frac{1}{2}\det(2)\det(\Delta_g)$ and compute $A_0^h$. You should arrive at (*1 point*)

$$A_0^h = \mathcal{V}\int_{\mathcal{M}_{T^2}} \mathrm{d}^2\tau \frac{\tau_2^{10}}{(2\pi)^{13}} \left(\det(\Delta_g)\right)^{-12} \det(2)\,, \tag{18}$$

where $\det(2)$ can be absorbed into a counterterm by modifying the action. The computation of $\det(\Delta_g)$ would lead to

$$\det\Delta_g = \tau_2^2 \mathrm{e}^{-\pi\tau_2/3}\Big|\prod_{n=1}^{\infty} 1 - \mathrm{e}^{2i\pi n\tau}\Big|^4. \tag{19}$$

plugging it into $A_0^h$ we arrive at the final result for the torus partition function

$$\begin{aligned} A_0^h &= \int_{\mathcal{M}_{T^2}} \frac{\mathrm{d}^2\tau}{2\pi\tau_2^2}(2\pi\tau_2)^{-12}\mathrm{e}^{4\pi\tau_2}\Big|\prod_{n=1}^{\infty} 1 - \mathrm{e}^{2i\pi n\tau}\Big|^{-48} \\ &= \frac{1}{2}\int_{\mathcal{F}_{\mathrm{PSL}(2,\mathbb{Z})}} \frac{\mathrm{d}^2\tau}{2\pi\tau_2^2}(2\pi\tau_2)^{-12}\mathrm{e}^{4\pi\tau_2}\Big|\prod_{n=1}^{\infty} 1 - \mathrm{e}^{2i\pi n\tau}\Big|^{-48}, \end{aligned} \tag{20}$$

where $\mathcal{F}_{\mathrm{PSL}(2,\mathbb{Z})}$ is the fundamental domain of the modular group $PSL(2,\mathbb{Z})$. Notice that integrating over $\mathcal{F}_{\mathrm{PSL}(2,\mathbb{Z})}$ leaves an unfixed residual gauge freedom given by the diffeomorphism $\sigma^1 \to -\sigma^1$, $\sigma^2 \to -\sigma^2$. Therefore a factor of 1/2 is necessary to remove the over-counting.