Physikalisches Institut
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Theoretische Physik

Exercise Sheet 3
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SS18

Exercises on Advanced Topics in String Theory

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More information at: `http://www.th.physik.uni-bonn.de/people/fierro/StringSS18/`

1 Open string scattering amplitudes (15 points)

Recall in the last exercise sheet of the winter term you evaluated the four tachyon scattering amplitude for the closed bosonic string on a certain Riemann surface (S^2), called the *Virasoro-Shapiro amplitude*. The aim of this exercise is to evaluate the same for the case of the open bosonic string on the D^2. This is known as the *Veneziano amplitude*.
As a reminder, to evaluate such scattering amplitudes one first employs the state-operator map to write the tachyon state in terms of a vertex operator (normal ordering is implicit):

$$V(p_i) \sim \int dx \; e^{ip_i \cdot X} = \int dx \; V(x, p_i) \; ,$$

where the integration is no longer over the entire worldsheet (as in case of the closed string vertex operators), but rather only on the boundary of the open string worldsheet. Then a general 4 point tachyon amplitude can be written as:

$$\mathcal{A}^{(4)} \sim \frac{g_s}{V_{CKG}} \int \prod_{i=1}^{4} dx_i \, \langle V(x_1, p_1) \ldots V(x_4, p_4) \rangle \; ,$$

where g_s is the string coupling constant and V_{CKG} is the volume of the conformal Killing group, which for the open string is $PSL(2, \mathbb{R})$ (cf. Blumenhagen et al). We can use the same trick as for the closed string case to remove infinite conformal Killing group factor i.e. introduce the c ghosts at the positions of any of the three tachyon vertex operators and omit the integration of the the boundary coordinates of those very vertex operators. Then, what we need to calculate in effect is:

$$\mathcal{A}^{(4)} \sim g_s \int dx_3 \, \langle \left(\prod_{i=1}^{3} c(x_i) \right) V(x_1, p_1) \ldots V(x_4, p_4) \rangle$$

1. Derive:
$$\langle \prod_{i=1}^{3} c(z_i) \rangle \sim z_{12} z_{23} z_{12} \; , \tag{1}$$
using general results of 2d CFTs. Here $z_{ij} = z_i - z_j$. (*1 point*)

2. Using the Wick theorem prove:

$$\langle V(x_1,p_1)\dots V(x_4,p_4)\rangle_{D^2} \sim \delta\left(\sum_{i=1}^{4} p_i\right)\prod_{j<l}|x_j - x_l|^{2\alpha' p_j\cdot p_l} \tag{2}$$

As compared to the closed string expression for such a correlation function, there is an extra factor of 2 appearing in the exponent. Can you think why? (*5 points*)

3. The $PSL(2,\mathbb{R})$ symmetry lets us fix any of the three coordinates to convenient positions. We choose: $x_4 \to \infty, x_1 \to 0, x_3 \to 1$ and rename $x_2 \to x$. Note that $x \in \mathbb{R}$ therefore operator ordering imposes $x \in \{0,1\}$.

   Using the previous results and the fact that the ghost system CFT and the boson CFT decouple, rewrite $\mathcal{A}^{(4)}$ in terms of the variable x only. Absorb all the overall factors into the normalisation as we would not be concerned with them for now. (*2 points*)

4. We introduce two definitions which will help simplify the expression you have obtained in the previous part:

   a) Mandelstam variables:

   $$s = -(p_1+p_2)^2 \ ; \ t = -(p_1+p_3)^2 \ ; \ u = -(p_1+p_4)^2 \ ,$$

   b) Euler beta function:

   $$B(a,b) = \int_0^1 dx\ x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

   Use this to arrive at the final expression for the Veneziano amplitude:

   $$\mathcal{A}^{(4)} \sim g_s(B(-\alpha' s-1,-\alpha' t-1)+B(-\alpha' s-1,-\alpha' u-1)+B(-\alpha' t-1,-\alpha' u-1))$$

   (*5 points*)

5. Fix t and find for what values of s the first $B(a,b)$ in the previous expression diverges. Can you see what all these poles correspond to?

   (*2 points*)