Exercises on Advanced Topics in String Theory

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Exercises Sheet 3

3.1 Supersymmetry on the worldsheet *(10 points)*

Consider the action

$$S = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma \left(\partial_\alpha X_\mu \partial^\alpha X^\mu + \overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right) \tag{1}$$

for the superstring. Here ρ^0 and ρ^1 form a representation for the two dimensional Clifford algebra and $\overline{\psi} = \psi^\dagger \beta = \psi^{\mathrm{T}}\beta$ with $\beta = \rho^0$ is the Dirac conjugate of ψ. $\psi^\mu = \begin{pmatrix}\psi^\mu_+ \\ \psi^\mu_-\end{pmatrix}$ is a two dimensional spinor (it transforms under the spinorial representation of the two dimensional Lorentz group). We choose the components of the spinor to be real i.e. $\psi^{\mu\,*}_+ = \psi^\mu_+$ and $\psi^{\mu\,*}_- = \psi^\mu_-$ since this is possible in two dimensions.

(a) Check that $\rho^0 = \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}$ and $\rho^1 = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$ satisfies the Clifford algebra and evaluate the signature of the worldsheet. (*2 points*)

(b) Show that the superstring action can be rewritten as

$$S = \frac{1}{\pi}\int \mathrm{d}^2\sigma \left(2\partial_+ X \partial_- X + i\psi_-\partial_+\psi_- + i\psi_+\partial_-\psi_+\right). \tag{2}$$

(*2 points*)

(c) Check that (1) is invariant under the global supersymmetry variations given by

$$\delta X^\mu = \bar{\epsilon}\psi^\mu, \quad \delta\psi^\mu = \rho^\alpha \partial_\alpha X^\mu \epsilon, \tag{3}$$

where ϵ is a Majorana spinor. (*2 points*)

(d) Show that these induce a conserved supercurrent

$$j_+ = \psi^\mu_+ \partial_+ X_\mu, \quad j_- = \psi^\mu_- \partial_- X_\mu. \tag{4}$$

(*2 points*)

(e) Show that the non-zero elements of the energy momentum tensor are given by

$$T_{++} = \partial_+ X_\mu \partial_+ X^\mu + \frac{i}{2}\psi_+^\mu \partial_+ \psi_{+\,\mu}, \quad T_{--} = \partial_- X_\mu \partial_- X^\mu + \frac{i}{2}\psi_-^\mu \partial_- \psi_{-\,\mu}. \tag{5}$$

(2 points)

3.2 Spinors in various dimensions *(10 points)*

Let us define the Clifford algebra in d dimensions by

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{6}$$

where $\mu, \nu = 0, 1, ..., d-1$. It holds that $\gamma^{\mu\,\dagger} = \gamma^\mu$ for Euclidean signature, whereas $\gamma^{\mu\,\dagger} = \gamma^0\gamma^\mu\gamma^0$ for Minkowski space. The 2^{d+1} matrices $\pm 1, \pm\gamma^\mu, \pm\gamma^{\mu\nu}, ...$ generate a finite group. Schur's Lemma states that an operator which commutes with all elements of a representation must be a multiple of the unitary element. In addition we introduce the anti-symmetrized products

$$\gamma^{\mu_1 \cdots \mu_p} = \frac{1}{p!}\left(\gamma^{\mu_1}\gamma^{\mu_2}...\gamma^{\mu_p} \pm \text{permutations}\right) \tag{7}$$

and for even dimensions $d = 2n$ the chirality operator

$$\gamma_{d+1} = \alpha\gamma^0\gamma^1...\gamma^{d-1}. \tag{8}$$

The charge conjugation matrix is defined via

$$(\gamma^\mu)^{\mathrm{T}} = \mp C_\pm \gamma^\mu C_\pm^{-1}. \tag{9}$$

One can show that both $C_\pm^{\mathrm{T}}$ exist in even dimensions and at least one of them exists in odd dimensions. In particular one has for $d = 2n$

$$C_\pm^{\mathrm{T}} = (-1)^{\frac{1}{2}n(n\pm 1)} C_\pm \tag{10}$$

and for $d = 2n + 1$

$$C = \begin{cases} C_+ & \text{for}\, n\, \text{odd} \\ C_- & \text{for}\, n\, \text{even} \end{cases}. \tag{11}$$

(a) Determine α s.t $\gamma_d^2 = 1$ holds and show that $\gamma_{d+1}^\dagger = \gamma_{d+1}$. *(1 point)*

(b) Using Schur's Lemma show that $C^{\mathrm{T}} = \pm C^{\mathrm{T}}$. *(1 point)*

(c) Show that the matrices $T^{\mu\nu} = -\frac{i}{2}\gamma^{\mu\nu}$ satisfy

$$[T^{\mu\nu}, T^{\rho\sigma}] = i\left(\eta^{\mu\rho}T^{\nu\sigma} + \eta^{\nu\sigma}T^{\mu\rho} + \eta^{\mu\sigma}T^{\nu\rho} - \eta^{\nu\rho}T^{\mu\sigma}\right). \tag{12}$$

(1 point)

A representation for which there is a matrix R s.t.

$$-(T^{\mu\nu})^* = RT^{\mu\nu}R^{-1} \tag{13}$$

is called a (pseudo-) real and complex otherwise. In particular one can show that $R^{\mathrm{T}} = \pm R$. A representation with a positive sign is called real, whereas the representation with the minus sign is called pseudoreal.

(d) Show that in the even-dimensional Euclidean case $d = 2n$

$$(T_{\pm}^{\mu\nu})^* = \begin{cases} -(C_{\pm})T_{\mp}^{\mu\nu}(C_{\pm})^{-1} & \text{for}\, n\, \text{odd} \\ -(C_{\pm})T_{\pm}^{\mu\nu}(C_{\pm})^{-1} & \text{for}\, n\, \text{even} \end{cases}, \tag{14}$$

where $T_{\pm}^{\mu\nu} = T^{\mu\nu}\frac{1}{2}(1 \pm \gamma_{d+1})$ are the generators associated to the respective chiral subspaces. (*1 point*)

(e) Evaluate for which dimensions the representations are real, pseudoreal and complex. You should find that the result only depends on the dimension mod 4. (*1 point*)

(f) We define in Euclidean signature

$$b_i^{\pm} = \frac{1}{2}(\gamma^{2i} \pm i\gamma^{2i+1}). \tag{15}$$

Show that

$$b_i^{\pm} = b_i^{\mp\,\dagger}, \quad \{b_i^{\pm}, b_j^{\mp}\} = \delta_{ij}, \quad \{b_i^+, b_j^+\} = \{b_i^-, b_j^-\} = 0. \tag{16}$$

(*2 points*)

(g) Let the highest weight state $|\Omega\rangle$ be defined by $b^i|\Omega\rangle = 0$ such that $|\Omega\rangle = |\frac{1}{2}, ..., \frac{1}{2}\rangle$. All other states are given by $|\pm\frac{1}{2}, ..., \pm\frac{1}{2}\rangle$. Show that this representation is reducible and decomposes into irreducible representations given by positive and negative chirality spinors respectively. (*2 points*)

(h) Show that $d = 8$ is special in the sense that the spinorial representations have the same dimension as the vector representation. There is a symmetry relation these representations, called triality symmetry. Can you guess this symmetry by inspecting the Dynkin diagram of $SO(8)$? (*1 point*)