Physikalisches Institut | Exercise 5
Universität Bonn | 07 May 2018
Theoretische Physik | SS 2018

# Exercises on Advanced Topics in String Theory

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http://www.th.physik.uni-bonn.de/people/fierro/StringSS18/
Due to: 14.05.2015

*"Todas las teorías son legítimas y ninguna tiene importancia. Lo que importa es lo que se hace con ellas."*

Jorge Luis Borges

## 1 Mini-supergravity

The purpose of this exercise is to consider a simple theory, where **supersymmetry is local**, i.e. where the infinitesimal Grassmann $\epsilon$ parameter depends on spacetime. For simplicity, we borrow the (0+1) worldline gravity theory, which describes a particle propagating on a $D$-dimensional spacetime. Here $x : I \to x(I) \simeq \mathbb{R}^{1,D-1}$, with supersymmetric partner given by $\psi$, $e$ is an auxilary field (einbein) and its fermionic partner we denote it by $\chi$. The action takes the form

$$S_p|_{SUGRA} = \int_I d\tau \left( \frac{1}{2e} \dot{x}^\mu \dot{x}_\mu + \frac{i}{e} \dot{x}^\mu \psi_\mu \chi - i \psi^\mu \dot{\psi}_\mu \right) . \tag{1}$$

(a) Show that (1) is invariant under reparametrizations, i.e. under the following infinitesimal transformations with parameter $\xi(\tau)$ given by *(4 points)*

$$\delta x^\mu = \xi \dot{x}^\mu \,, \quad \delta \psi^\mu = \xi \dot{\psi}^\mu \,, \quad \delta e = \frac{d}{d\tau}(\xi e) \,, \quad \delta \chi = \frac{d}{d\tau}(\xi \chi) \,. \tag{2}$$

(b) Show that (1) is invariant under local supersymmetry transformations, which read *(4 points)*

$$\delta x^\mu = i \epsilon \psi^\mu \,, \quad \delta \psi^\mu = \frac{1}{2e} \Big( \dot{x}^\mu - i \chi \psi^\mu \Big) \epsilon \,, \quad \delta e = -i \chi \epsilon \,, \quad \delta \chi = \dot{\epsilon} \,. \tag{3}$$

(c) Show that in the gauge $e = 1, \chi = 0$, one obtains the following action *(2 points)*

$$S_p|_{SUSY} = \int d\tau \left( \frac{1}{2} \dot{x}^\mu \dot{x}_\mu - i \psi^\mu \dot{\psi}_\mu \right) . \tag{4}$$

Moreover show that the action is invariant under global supersymmetry transformations

$$\delta x^\mu = i \epsilon \psi^\mu \,, \quad \delta \psi^\mu = \frac{1}{2} \epsilon \dot{x}^\mu \,. \tag{5}$$

Here $\epsilon$ is now an infinitesimal real constant Grassmann parameter.

(d) Derive the constraint equation for the gauge-fixed theory, i.e. the e.o.m. for $e$ and $\chi$. *(4 points)*

# 2 SCFT & superconformal algebra

In Supergravity theories, the vielbein formalism is a necessary ingredient to describe spinors on a curved manifold $M$ endowed with metric $g$. We define the vielbein as the $\{e_a^\mu\} \in GL(dimM, \mathbb{R})$ such that

$$e_a^\mu e_b^\nu g_{\mu\nu} = \eta_{ab} \,. \tag{6}$$

The inverse is given by the $\{e_\mu^a\} \in GL(dimM, \mathbb{R})$ such that $e_\mu^a e_a^\nu = \delta_\mu^\nu$ and $e_\mu^a e_b^\mu = \delta_b^a$.

(a) Verify the identity, $e_a^\mu e_b^\nu \eta^{ab} = g^{\mu\nu}$. *(0.25 points)*

(b) Let $\gamma^a$ be the Dirac matrices in Minkowski spacetime, which satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$. Denote the curved spacetime counterparts of the Dirac matrices by $\gamma^\mu := e_a^\mu \gamma^a$. Show that *(1.75 points)*

$$\{\gamma^\mu, \gamma^\mu\} = 2g^{\mu\nu} \,. \tag{7}$$

Now we turn back to the superstring theory action given by the supersymmetry extension of the Polyakov action. This is given by a 2d $N = 1$ supersymmetry with a scalar multiplet $(X^\mu, \psi^\mu, F^\mu)$ and the supergravity multiplet $(e_\alpha^a, \chi_\alpha, A)$. The complete action invariant under local supersymmetry transformations reads

$$S_P|_{SUGRA} = -\frac{1}{8\pi}\int_\Sigma d^2\sigma e\left(\frac{2}{\alpha'}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu + 2i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu - i\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu \left(\sqrt{2}\alpha' \partial_\beta X_\mu - \frac{i}{4}\bar{\chi}_\beta \psi\right)\right).$$

Here $e = |\det e_\alpha^a| = \sqrt{-h}$. This action is invariant under the worldsheet symmetries:

- Local supersymmetry
- Weyl transformations
- Super-Weyl transformations
- 2d Lorentz transformations
- Reparametrizations

Similarly as in the bosonic string, we can make use of the worldsheet symmetries to gauge away the degrees of freedom given by the worldsheet metric and the worldsheet gravitino, i.e. we fix a **superconformal gauge** locally. At the end of the day our action reduces to

$$S_P|_{SUGRA} \xrightarrow{\text{sup. conf. gauge}} S_P|_{SCFT} = \tfrac{1}{4\pi}\int d^2z\left(\tfrac{2}{\alpha'}\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu\right). \tag{8}$$

For theories with fermions, the energy momentum tensor $T$ and its fermionic supersymmetry counterpart $T_F$ (supercurrent) are defined by

$$T_{\alpha\beta} = \frac{2\pi}{e}\frac{\delta S}{\delta e_a^\beta} e_{\alpha a} \,, \quad T_{F_\alpha} = \frac{2\pi}{e}\frac{\delta S}{\partial \bar{\chi}^\alpha} \,. \tag{9}$$

(c) Show that the worldsheet supercurrents are given by *(4 points)*

$$T_F(z) = i\sqrt{\frac{2}{\alpha'}}\psi^\mu(z)\partial X_\mu(z)\,, \quad \tilde{T}_F(\bar{z}) = i\sqrt{\frac{2}{\alpha'}}\tilde{\psi}^\mu(\bar{z})\bar{\partial}X_\mu(\bar{z})\,. \tag{10}$$

Moreover, show that the holomorphic energy momentum tensor $T(z)$ is given by

$$T(z) = -\frac{1}{\alpha'}\partial X^\mu(z)\partial X_\mu(z) - \frac{1}{2}\psi^\mu(z)\partial\psi_\mu(z)\,. \tag{11}$$

You might recall from the previous course that $X^\mu(z)X^\nu(w) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(z-w)$. Similarly, it is not hard to obtain the OPEs of a free fermion theory, which read

$$\psi^\mu(z)\psi^\nu(w) \sim \frac{\eta^{\mu\nu}}{z-w}\,, \quad \tilde{\psi}^\mu(\bar{z})\tilde{\psi}^\nu(\bar{w}) \sim \frac{\eta^{\mu\nu}}{\bar{z}-\bar{w}}\,. \tag{12}$$

The rest of the OPEs among $X^\mu$ and $\psi^\nu$ are trivial.

(d) Show that $\psi^\mu(z)$ and $T_F(z)$ are primary fields of conformal weight $(h,\bar{h})$: $(\frac{1}{2},0)$ and $(\frac{3}{2},0)$ respectively. *(3 points)*

(e) Show that the generators given by *(4 points)*

$$T_{F_\epsilon} = \oint \frac{dz}{2\pi i}\epsilon(z)T_F(z)\,, \tag{13}$$

where $\epsilon(z)$ is an anticommutating infinitesimal parameter, generate the superconformal transformations

$$\delta_\epsilon X^\mu(z) = -i\sqrt{\frac{2}{\alpha'}}\epsilon(z)\psi^\mu(z)\,, \quad \delta_\epsilon\psi^\mu(z) = i\sqrt{\frac{2}{\alpha'}}\epsilon(z)\partial X^\mu(z)\,. \tag{14}$$

*Hint:* $\delta_\epsilon\phi(z) = -[T_{F_\epsilon}, \phi(z)]$.

(f) Show that the commutator of two superconformal transformations is a conformal transformation, i.e. *(4 points)*

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_\xi\,, \quad \delta_\xi = \frac{1}{2}\epsilon_1\epsilon_2\,. \tag{15}$$

Similarly, the commutator of a conformal transformation and a superconformal transformation is a superconformal transformation. The conformal transformations thus close to form the **superconformal algebra**. The $N=1$ superconformal algebra in OPE form is given by

$$T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{16}$$

$$T(z)T_F(w) \sim \frac{\frac{3}{2}T_F(w)}{(z-w)^2} + \frac{\partial T_F(w)}{z-w}\,, \tag{17}$$

$$T_F(z)T_F(w) \sim \frac{\frac{2c}{3}}{(z-w)^3} + \frac{2T(z)}{z-w}\,. \tag{18}$$

Here $N=1$ refers to the number of $(\frac{3}{2},0)$ currents. In the present case there is also an antiholomorphic copy of the same algebra, so we have an $(N,\bar{N}) = (1,1)$ **superconformal field theory (SCFT)**.

(g) Using the explicit form of the supercurrent and the energy momentum tensor in (10) and (11), verify the OPEs (16), (17) and (18). You should find out that $\mathrm{c} = \frac{3}{2}D$ here. *(6 points)*

Denote for the following discussion $w$ as the cylindrical coordinate $w = \sigma^1 + i\sigma^2$. For the closed string $w \sim w + 2\pi$. Lorentz invariance allows two possible periodicity conditions for $\psi^\mu(\tilde{\psi}^\mu)$

- Ramond (**R**): $\psi^\mu(w + 2\pi) = +\psi^\mu\,,$
- Neveu-Schwarz (**NS**): $\psi^\mu(w + 2\pi) = -\psi^\mu\,.$

We can rewrite this as $\psi^\mu(w + 2\pi) = e^{i2\pi a}\psi^\mu(w)$, where $a \in \{0, \frac{1}{2}\}$. Going back to the conformal plane, the Laurent expansions of the fermionic fields are given by

$$\psi^\mu(z) = \sum_{r\in\mathbb{Z}+a} \frac{\psi^\mu_r}{z^{r+\frac{1}{2}}}\,, \quad \tilde{\psi}^\mu(\bar{z}) = \sum_{r\in\mathbb{Z}+a} \frac{\tilde{\psi}^\mu_r}{\bar{z}^{r+\frac{1}{2}}}\,. \tag{19}$$

(h) Show that $\{\psi^\mu_r, \psi^\nu_s\} = \{\tilde{\psi}^\mu_r, \tilde{\psi}^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0}\,.$ *(4 points)*

For $T$ and $T_F$ the Laurent expansions are

$$T_F(z) = \sum_{r\in\mathbb{Z}+a} \frac{G_r}{z^{r+\frac{3}{2}}}\,, \quad T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}\,. \tag{20}$$

(i) Use the CFT countour calculations to obtain the superconformal algebra commutation relations in terms of the modes, which is given by *(6 points)*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\mathrm{c}}{12}(m^3 - m)\delta_{m+n,0}\,, \tag{21}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{\mathrm{c}}{12}(4r^2 - 1)\delta_{r+s,0} \tag{22}$$

$$[L_m, G_r] = \frac{m-2r}{2}G_{m+r}\,. \tag{23}$$

Mathematically the relations (21), (22) and (23) determine an infinite-dimensional super Lie algebra, A finite algebra contained in the infinite-dimensional one is generated by $\{L_0, L_{\pm 1}, G_{\pm\frac{1}{2}}\}$ and turns out to be $\mathfrak{osp}(1|2)$. The corresponding super group $OSP(1|2)$ plays the same role for SCFTs as $SL(2,\mathbb{C})/\mathbb{Z}_2$ does for the usual CFTs.