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Exercise 7
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Exercises on Advanced Topics in String Theory

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http://www.th.physik.uni-bonn.de/people/fierro/StringSS18/

–Exercises 7–

7.1 Spin structures and GSO projection *(25 points)*

In this exercise we want to investigate closed superstring theories by analysing their one-loop partition function. Consider all string states $|\mathrm{st.}\rangle$ living in the Hilbert space $\mathcal{H}_{\mathrm{cl.}}$ of the theory, then the partition function is given by the following trace over the states

$$Z(\tau) = \sum_{|\mathrm{st.}\rangle \in \mathcal{H}_{\mathrm{cl.}}} \langle \mathrm{st.}|\mathrm{e}^{-\tau_2 \ell H} \mathrm{e}^{i\tau_1 P}|\mathrm{st.}\rangle, \tag{1}$$

where H is the Hamiltonian and P is the momentum operator. $\tau\ell = (\tau_1 + i\tau_2)\ell$ is the complex structure of the torus and ℓ is the string length. The Hamiltonian for the closed strings is a sum over a Hamiltonian for the left- and righthanded states

$$H = H_L + H_R. \tag{2}$$

The Hamiltonians for (lefthanded) stringstates in the i-th direction of the target space from the Neveu-Schwarz and Ramond sector are given by

$$H_{\mathrm{NS},L} = \frac{2\pi}{\ell}\left[\sum_{r=0}^{\infty}\left(r+\frac{1}{2}\right)\psi^i_{-r-1/2}\psi^i_{r+1/2} - \frac{1}{48}\right], \tag{3}$$

$$H_{\mathrm{R},L} = \frac{\pi}{\ell}\left[\sum_{r=0}^{\infty} r\psi^i_{-r}\psi^i_{r} + \frac{1}{3}\right]. \tag{4}$$

For the righthanded sector the Hamiltonian look the same except that the oscillators carry a "~" The the momentum operator is given by $P = \frac{2\pi}{\ell}(N - \tilde{N})$, where $N = \sum_{r=0}^{\infty}\left(r+\frac{1}{2}\right)\psi^i_{-r-1/2}\psi^i_{r+1/2}$ ($\sum_{r=0}^{\infty} r\psi^i_{-r}\psi^i_{r}$) is the number operator in the Neveu-Schwarz (Ramond) sector.

(a) Show that the closed string partition function $Z(\tau)$ splits into a product of a left- and righthanded partition function

$$Z(\tau) = \mathrm{Tr}_{\mathcal{H}_L} q^{N+E_0} \mathrm{Tr}_{\mathcal{H}_R} \bar{q}^{\tilde{N}+\tilde{E}_0}, \tag{5}$$

where the trace is taken over the Hilbert space $\mathcal{H}_L$ ($\mathcal{H}_R$) of left- (right-) handed string states and E_0 is the zero point energy. *(2 points)*

(b) Show that the partition function for the Neveu-Schwarz sector is given by

$$\mathrm{Tr}_{\mathcal{H}_{\rm NS}} q^{N_{\rm NS}+E_{0,\rm NS}} = \frac{\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^4}{\eta^4} \tag{6}$$

and for the Ramond sector is given by

$$\mathrm{Tr}_{\mathcal{H}_{\rm R}} q^{N_{\rm R}+E_{0,\rm R}} = \frac{\vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}^4}{\eta^4}, \tag{7}$$

where $q = \mathrm{e}^{2\pi i \tau}$. *(5 points)*

(c) Construct the partition function with all possible closed strings by glueing left- and righthanded states together i.e. make use of (**??**). Check whether the partition function is modular invariant. *(3 points)*

Something must be wrong in the way we constructed the closed string theory above. To cure the problem let us investigate boundary conditions for fermions on the worldsheet more carefully.

(d) Consider a string worldsheet with the topology of a sphere on which bosonic and fermionic states live. The rotation group on the manifold is $SO(2)$ which is isomorphic to $U(1)$. The representations of $SO(2)$ are given by specifying the eigenvalues of the only generator W of $SO(2)$. Show that a vector field V^μ decomposes into the components V^+ and V^- of $W = 1$ and $W = -1$, respectively. A two-components spinor field ψ^A has components ψ^+ and ψ^- with $W = 1/2$ and $W = -1/2$, respectively. How many degrees of freedom does ψ^A have if it is Majorana? *(1,5 points)*

(e) Consider V^+ parallel transported around a closed path γ on the sphere. V^+ will then pick up a phase $\mathrm{e}^{i\alpha}$. What happens to ψ^+ when parallel transported around γ? Explain why one could naively assume there would be a sign ambiguity for the parallel transported ψ^+? *(1,5 points)*

(f) Now shrink the path γ to a point and show that the sign must be $+1$. What happens if γ enclosed a handle on the worldsheet[1]? The different signs are called *Spin structures.* *(1 point)*

(g) Determine all Spin structures on a two dimensional torus. Parameterise the torus by the worldsheet coordinates τ and σ and identify the boundary conditions for the Neveu-Schwarz and Ramond sector. *(2 points)*

To incorporate the Spin structure in τ direction, one introduces the GSO-operator $(-1)^F$. It anticommutes with fermions

$$(-1)^F \psi^i_\nu = -\psi^i_\nu (-1)^F \tag{8}$$

[1] Of course this is not possible for a sphere but a higher genus worldsheet.

and string states with odd (even) number of fermionic oscillators have GSO eigenvalue -1 $(+1)$. For the Neveu-Schwarz sector the fermion number operator F is given by

$$F = \sum_{r=0}^{\infty} \psi^i_{-r-1/2}\psi^i_{r+1/2} \tag{9}$$

and for the Ramond sector the GSO operator takes the form

$$(-1)^F = \pm\Gamma^{11} \cdot (-1)^{\sum_{r\geq 1}^{\infty} \psi^i_{-r}\psi^i_r}, \tag{10}$$

with $\Gamma^{11} = \Gamma^0\Gamma^1 ... \Gamma^9$ the ten dimensional chirality operator.

(h) Calculate the partition functions for the left- and righthanded Ramond and Neveu-Schwarz sector with the insertion of the GSO-projection $\frac{1\pm(-1)^F}{2}$. For the Neveu-Schwarz sector the GSO-projection takes in the left- and righthanded sector the form $\frac{1-(-1)^F}{2}$

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{NS},L}}\left[q^{N_{\mathrm{NS}}+E_{0,\mathrm{NS}}}\frac{1-(-1)^F}{2}\right], \tag{11}$$

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{NS},R}}\left[q^{\tilde{N}_{\mathrm{NS}}+\tilde{E}_{0,\mathrm{NS}}}\frac{1-(-1)^F}{2}\right]. \tag{12}$$

In the lefthanded Ramond sector the GSO projection takes the same form as in the Neveu-Schwarz sector

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{R},L}}\left[q^{N_{\mathrm{R}}+E_{0,\mathrm{R}}}\frac{1-(-1)^F}{2}\right], \tag{13}$$

but in the righthanded Ramond sector we have the possibility to choose the sign in the GSO-projection

$$A \quad \mathrm{Tr}_{\mathcal{H}_{\mathrm{R},R}}\left[q^{\tilde{N}_{\mathrm{R}}+\tilde{E}_{0,\mathrm{R}}}\frac{1+(-1)^F}{2}\right], \tag{14}$$

$$B \quad \mathrm{Tr}_{\mathcal{H}_{\mathrm{R},R}}\left[q^{\tilde{N}_{\mathrm{R}}+\tilde{E}_{0,\mathrm{R}}}\frac{1-(-1)^F}{2}\right]. \tag{15}$$

Identify the spin structures for the different pieces of the partition function. The possibility to choose the GSO projetion in the righthanded Ramond sector leads to two different closed string theories called Type IIA and IIB. (*6 points*)

(i) Count the fermionic and bosonic degrees of freedom for the tachyonic, massless and first massive states and comment your result. Show that the one-loop partition function vanishes for Type IIA and IIB. (*3 points*)