Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise Sheet 1
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Exercises Superstring Theory

Priv.-Doz. Stefan Förste, Cesar Fierro, Urmi Ninad

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http://www.th.physik.uni-bonn.de/people/fierro/StringWS1718/

1 The relativistic point particle

Consider the action for a free relativistic point particle of mass m and coordinates $x^\mu(\tau)$ on d-dimensional Minkowski spacetime M, where $\mu = 0, ..., d-1$, and τ is an arbitrary parameter along the worldline of the particle,

$$S_{pp} = -m \int ds = -m \int d\tau \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \,. \tag{1}$$

(a) Show that the action (1) is invariant under Poincaré transformations. (*0.5 Point*)

(b) Show that the action (1) is invariant under worldline reparametrizations $\tau \to \tau'(\tau)$. (*1 Point*)

(c) Show that

$$p^\mu = \frac{m\dot{x}^\mu}{\sqrt{-\eta_{\nu\rho}\dot{x}^\nu \dot{x}^\rho}} \tag{2}$$

is a conserved quantity by evaluating the Euler-Lagrange equations and also by exploiting the symmetry $x^\mu \to x^\mu + b^\mu$. (*1 Point*)

(d) Why is this action inappropriate to describe massless particles? (*0.5 Point*)

In order to describe massless particles and avoid the square root in the action (1), we introduce an auxiliary field $e(\tau)$ and write the action S_e below

$$S_e = -\frac{1}{2}\int d\tau e\left(-\frac{1}{e^2}\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} + m^2\right) \,. \tag{3}$$

(e) Show that the action (3) is equivalent to (1). *Hint: Integrate out e.* (*1 Point*)

(f) Explain the statement:"the particle is coupled to worldline gravity". What kind of field is e? (*1 Point*)

(g) Show the invariance of S_e under reparametrizations of τ. How does e transform? (*1 Point*)

2 The Nambu-Goto action versus the Polyakov action

We now generalize the action of a point particle, given by (1), to the action of a one-dimensional object, a string. Now, instead of a worldline, the string will sweep out a worldsheet Σ. The action for such a string is called *Nambu-Goto action* and it is given by

$$S_{NG} = -T\int_\Sigma d^2\sigma \sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu})} \,. \tag{4}$$

Here $\sigma^\alpha = (\sigma, \tau)$ label the coordinates on the worldsheet, and $X^\mu(\sigma,\tau)$, $\mu = 0, ..., d-1$, are maps of the worldsheet Σ into d-dimensional Minkowski spacetime M.

(a) Write down explicitly the action (4), i.e. without refering to α, but to 0 ($\equiv \sigma$) and 1 ($\equiv \tau$) instead. (*1 Point*)

We now introduce an auxiliary field, a metric $h_{\alpha\beta}(\sigma,\tau)$ on the worldsheet with signature $(-,+)$ (similarly to the introduction of $e(\tau)$ in the action of a point particle, cf. Exercise 1) to avoid problems related to the square root. This new action is called *Polyakov action* and it is given by

$$S_P = -\frac{T}{2}\int d^2\sigma \sqrt{-h}(h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}) \ , \tag{5}$$

where $h = \det h_{\alpha\beta}$.

(b) Show that the Polyakov action (5) is equivalent to the Nambu-Goto action (4). (*3 Point*)
Hint: Integrate out $h_{\alpha\beta}$.

3 Symmetries of the Polyakov action

- **Global symmetries of the Polyakov action** *1.5 Point*

(a) Show that the Polyakov action is invariant under Poincaré transformations. (*0.5 Point*)

(b) Evaluate the corresponding conserved currents using the Noether procedure, which we recall briefly below. (*1 Point*)

If the Lagrangian is invariant under an infinitesimal transformation of the fields given by

$$\phi^a \to \phi^a + \delta\phi^a \ , \quad \delta\phi^a = \epsilon^i h_i^a(\phi^b) \ , \tag{6}$$

where ϵ^i is infinitesimal and h_i^a denotes a function of the fields ϕ^a, then the current j_i^α defined by

$$\epsilon^i j_i^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\delta\phi^a \tag{7}$$

is conserved[1].

The infinitesimal transformations for Poincaré transformations are given by

$$X^\mu \to X^\mu + \epsilon^\mu \ , \quad X_\mu \to \epsilon a_{\mu\nu} X^\nu \ , \quad a_{\mu\nu} = -a_{\nu\mu} \ . \tag{8}$$

Hint: Evaluate the currents using the fixed worldsheet metric given by (cf. "Local symmetries of the Polyakov action" below)

$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ . \tag{9}$$

- **Local symmetries of the Polyakov action** *2.5 Points*

(c) Show that the action is invariant under worldsheet reparametrizations $\sigma^\alpha \to \sigma'^\alpha(\sigma^\beta)$. (*0.5 Point*)

(d) Show that the action is invariant under Weyl transformations $h_{\alpha\beta} \to e^{\phi(\sigma^\alpha)} h_{\alpha\beta}$. (*0.5 Point*)

(e) Show that the local symmetries can be used to fix the worldsheet metric locally to

$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ . \tag{10}$$

Why is it not possible to use this metric globally? (*1.5 Point*)

[1] Note that the index "i" in equation (6) might be a multi-index.