Physikalisches Institut
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Theoretische Physik

Exercise Sheet 3
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Exercises Superstring Theory

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Course web page → `http://www.th.physik.uni-bonn.de/people/fierro/StringWS1718/`

1 Poisson brackets for the classical string

Recall from the lecture that the expansion modes for closed strings are given by

$$X_L^\mu(\sigma_+) = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu \sigma_+ + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in\sigma_+} \ , \tag{1}$$

$$X_R^\mu(\sigma_-) = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu \sigma_- + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-2in\sigma_-} \ . \tag{2}$$

The canonical momentum conjugate to X^μ is given by

$$P^\mu(\sigma,\tau) = \frac{\delta S}{\delta \dot{X}_\mu} = T\dot{X}^\mu \ , \tag{3}$$

and the classical Poisson brackets are

$$\{P^\mu(\sigma,\tau), P^\nu(\sigma',\tau)\} = 0, \quad \{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\} = 0, \quad \{X^\mu(\sigma,\tau), P^\nu(\sigma',\tau)\} = \eta^{\mu\nu}\delta(\sigma-\sigma') \ . \tag{4}$$

Show that the Poisson brackets for the modes are

$$\{\alpha_m^\mu, \alpha_n^\nu\} = \{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\} = -im\eta^{\mu\nu}\delta_{m+n,0} \ , \quad \{\alpha_m^\mu, \tilde{\alpha}_n^\nu\} = 0 \ , \tag{5}$$

$$\{x^\mu, x^\nu\} = \{p^\mu, p^\nu\} = 0 \ , \quad \{x^\mu, p^\nu\} = \eta^{\mu\nu} \ , \tag{6}$$

$$\{x^\mu, \tilde{\alpha}_n^\nu\} = \{p^\mu, \tilde{\alpha}_n^\nu\} = 0 = \{x^\mu, \alpha_n^\nu\} = \{p^\mu, \alpha_n^\nu\} \ . \tag{7}$$

Hint: Start by expressing α_m^μ and $\tilde{\alpha}_m^\mu$ as linear combinations of $X^\mu(\sigma,\tau)$ and $P^\mu(\sigma',\tau)$ with τ fixed.

You will also need the Fourier expansion of the Dirac delta function on the interval $[0,\pi]$

$$\delta(\sigma-\sigma') = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} e^{2in(\sigma-\sigma')} \ . \tag{8}$$

2 The Classical Virasoro Algebra

Consider the Polyakov action S_P given in Exercise sheet 1. The energy-momentum tensor of the world-sheet theory is given by

$$T_{\alpha\beta} = -\frac{2}{T\sqrt{-h}}\frac{\delta S_P}{\delta h^{\alpha\beta}} = 0 \ , \tag{9}$$

where the second equality comes from the fact that the energy-momentum tensor vanishes in 2d.

(a) Show that in light-cone coordinates ($\sigma_\pm = \tau \pm \sigma$) the components of the energy-momentum tensor are

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu = 0 \ , \tag{10}$$

$$T_{--} = \partial_- X^\mu \partial_- X_\mu = 0 \ , \tag{11}$$

$$T_{+-} = T_{-+} = 0 \ , \tag{12}$$

where (12) means that these components vanish identically due to tracelessness of the energy-momentum tensor.
(*1 Point*)

(b) Using the mode expansions for closed strings[1] – equations (1) and (2) –, show that the mode expansions for the energy-momentum tensor are given by (*1 Point*)

$$T_{--} = 2l_s^2 \sum_{m=-\infty}^{\infty} L_m e^{-2im(\tau-\sigma)} \ , \quad T_{++} = 2l_s^2 \sum_{m=-\infty}^{\infty} \tilde{L}_m e^{-2im(\tau+\sigma)} \ , \tag{13}$$

where the Fourier coefficients are the *Virasoro generators*

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n, \quad \tilde{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n, \tag{14}$$

(c) Show that the Virasoro generators satisfy the following algebra, called *classical Virasoro algebra*, (*0.5 Point*)

$$\{L_n, L_m\} = i(n-m)L_{n+m} \ . \tag{15}$$

The classical Virasoro algebra also appears due to the fact that the fixing of the worldsheet metric to

$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{16}$$

does not fix the diffeomorphism symmetry completely, but allows for reparametrizations of the light-cone coordinates, i.e.

$$\sigma_+ \to \sigma'_+(\sigma_+) \ , \quad \sigma_- \to \sigma'_-(\sigma_-) \ . \tag{17}$$

To see this you should follow the procedure below.

(d) Show that S_P is invariant under[2]

$$\delta X^\mu = a_n e^{2in\sigma_-} \partial_- X^\mu \ , \tag{18}$$

and that this gives rise to the corresponding conserved current (*1 Point*)

$$j = e^{2in\sigma_-} \partial_- X^\mu \partial_- X_\mu \ . \tag{19}$$

(e) Finally, show that the corresponding conserved charge

$$Q_n = \int d\sigma j \tag{20}$$

is proportional to the Virasoro generator L_n. (*0.5 Point*)

[1] One can get the result for the case of open strings using the same procedure adopted here.
[2] Note that this is only part of the symmetry, i.e., the action is also invariant under $\delta X^\mu = a_n e^{2in\sigma_+} \partial_+ X^\mu$.