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Exercise Sheet 5
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# Exercises Superstring Theory

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## 1 The Quantum Virasoro Algebra

In **Exercise Sheet 3** the classical Virasoro generators for closed strings were introduced as the conserved charges associated with reparametrizations of the worldsheet light-cone coordinates $\sigma^{\pm} \to \sigma^{\pm} + f_n(\sigma^{\pm})$. They are given by[1]

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n, \quad \tilde{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n. \tag{5.1}$$

Recall the commutation relations and the normal ordering of the operators in canonical quantization of the bosonic string from exercise **Exercise Sheet 4**. The *quantum* Virasoro generators must then be normal-ordered, i.e.,

$$\hat{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} : \alpha_{m-n} \cdot \alpha_n : \ , \quad \hat{\tilde{L}}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} : \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n : . \tag{5.2}$$

The goal of this exercise is to obtain the **quantum Virasoro algebra**

$$[\hat{L}_m, \hat{L}_n] = \ ?, \qquad [\hat{\tilde{L}}_m, \hat{\tilde{L}}_n] = \ ?, \qquad [\hat{L}_m, \hat{\tilde{L}}_n] = \ ? \tag{5.3}$$

For brevity, let us now neglect the hat $\hat{}$ operator symbol and deal with only one set of these generators, say $L_m$.

(a) Show that *(1 point)*

$$[L_m, \alpha_n^{\mu}] = -n\alpha_{m+n}^{\mu} \ . \tag{5.4}$$

(b) Show that the normal ordered expression for $L_m$ is $(m \neq 0)$ *(1 point)*

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{-1} \alpha_n \cdot \alpha_{m-n} + \frac{1}{2} \sum_{n=0}^{\infty} \alpha_{m-n} \cdot \alpha_n \ . \tag{5.5}$$

(c) Use the results from (a) and (b) to show that *(1 point)*

$$\begin{aligned} [L_m, L_n] \ &= \ \frac{1}{2} \sum_{p=-\infty}^{0} \{(m-p)\alpha_p \cdot \alpha_{m+n-p} + p\alpha_{n+p} \cdot \alpha_{m-p}\} \\ &\quad + \frac{1}{2} \sum_{p=1}^{\infty} \{(m-p)\alpha_{m+n-p} \cdot \alpha_p + p\alpha_{m-p} \cdot \alpha_{n+p}\} \ . \end{aligned} \tag{5.6}$$

[1] The $\tilde{L}_m$ are, of course, absent for the open string.

(d) Change the summation variable in the second and fourth terms of (5.6) and assume $n > 0$ to rewrite (*1.5 point*)

$$\begin{aligned}[L_m, L_n] \;&=\; \frac{1}{2}\left\{\sum_{q=-\infty}^{0}(m-n)\alpha_q\cdot\alpha_{m+n-q}+\sum_{q=1}^{n}(q-n)\alpha_q\cdot\alpha_{m+n-q}\right\}\\ &+\frac{1}{2}\left\{\sum_{q=n+1}^{\infty}(m-n)\alpha_{m+n-q}\cdot\alpha_q+\sum_{q=1}^{n}(m-q)\alpha_{m+n-q}\cdot\alpha_q\right\}\ .\end{aligned} \tag{5.7}$$

Is this expression normal-ordered?

(e) For $m + n \neq 0$, show that (*2 points*)

$$[L_m, L_n] = (m-n)L_{m+n}\ . \tag{5.8}$$

(f) For $m + n = 0$, show that (*3.5 points*)

$$[L_m, L_{-m}] = 2mL_0 + \frac{d}{12}m(m^2-1)\ , \tag{5.9}$$

where $d$ is the number of spacetime dimensions of the target Minkowski spacetime.

*Hint: Normal order the term which was not normal-ordered in part (d). The following relation might be useful*

$$\sum_{k=1}^{m} k^2 = \frac{1}{6}(2m^3+3m^2+m)\ .$$

Combining the results from (e) and (f), you are able to see that the *quantum Virasoro algebra* is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\ , \tag{5.10}$$

where $c = d$ is called the *central charge* and equals the number of spacetime dimensions, as already stated in part (f). The term proportional to the central charge $c$ is called *central extension* and it arises exclusively as a quantum effect ($\equiv$ it is absent in the classical theory).

From part (b), you should note that, in particular, the quantum normal-ordered version of $L_0$ becomes

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n\ . \tag{5.11}$$

Indeed, this is the only Virasoro generator for which normal ordering matters, i.e. $L_0$ is not completely determined by its classical expression. Since an arbitrary constant could have appeared in this expression, one should add a constant $a$ to $L_0$ in all formulas. In other words, one has

$$L_0 \ \to L_0 + a\ . \tag{5.12}$$

Recall from **Exercise Sheet 1 & 3** that the classical constraints (vanishing of the energy-momentum tensor $T_{++} = 0$ and $T_{--} = 0$) imply $L_m = 0$, $\forall m$. This cannot be implemented in the quantum theory anymore (otherwise it would violate the quantum Virasoro algebra). Therefore, a *physical state* $|\phi\rangle$ in the quantum theory is defined as a state that is annihilated by half of the Virasoro generators which also satisfies the mass-shell condition, i.e.,

$$\begin{aligned}&L_m\,|\phi\rangle = 0\ , \qquad m > 0\ ,\\ &(L_0+a)\,|\phi\rangle = 0\ .\end{aligned} \tag{5.13}$$

## H5.2 Lorentz symmetry & the bosonic string spectrum

*(5 points)*

In **Exercise Sheet 1**, you obtained the conserved currents associated with invariance under Poincaré transformations of $X^\mu$, which in the conformal gauge read

$$\begin{aligned} P^\mu_\alpha &= -T\partial_\alpha X^\mu \ , \\ j^{\mu\nu}_\alpha &= -T(X^\mu \partial_\alpha X^\nu - X^\nu \partial_\alpha X^\mu) \ , \end{aligned}$$

where the first current is associated with invariance under translations and the second one is associated with invariance under Lorentz transformations.

(a) Using the oscillator mode expansions of the classical closed bosonic string

$$\begin{aligned} X^\mu_L(\sigma^+) &= \frac{1}{2}(x^\mu + c^\mu) + \frac{\pi\alpha'}{l}p^\mu\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-\frac{2\pi}{l}in\sigma^+} \ , \\ X^\mu_R(\sigma^-) &= \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{l}p^\mu\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-\frac{2\pi}{l}in\sigma^-} \ , \end{aligned} \tag{1}$$

show that the total conserved charge associated with invariance under Lorentz transformations (total angular momentum) for the classical closed bosonic string, can be expressed as *(1 point)*

$$-J^{\mu\nu}_0 = -\int d\sigma j^{\mu\nu}_0 \ = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n) - i\sum_{n=1}^{\infty}\frac{1}{n}(\tilde{\alpha}^\mu_{-n}\tilde{\alpha}^\nu_n - \tilde{\alpha}^\nu_{-n}\tilde{\alpha}^\mu_n) \ . \tag{5.14}$$

Note that, fortunately, these classical Lorentz generators can also be interpreted as quantum Lorentz generators $\hat{J}^{\mu\nu}_0$ with the same expression (5.14) since there are no normal-ordering ambiguities.

(b) Show that *(3 points)*

$$[\hat{L}_m, \hat{J}^{\mu\nu}_0] = 0 \ . \tag{5.15}$$

*Hint: Use (5.4) and calculate separately the commutator of $\hat{J}^{\mu\nu}_0$ with the zero-mode part of $\hat{L}_m$ and the commutator of $\hat{J}^{\mu\nu}_0$ with the oscillator part of $\tilde{L}_m$. In order to compare the result from the oscillator part with the zero-mode part, do the following:*

*i) in the commutator for the oscillator part, change some summation indices $n \to -n$ for terms with $\alpha^\mu_{-n}$,*

*ii) permute the resulting terms to obtain commutators that will cancel out with each other,*

*iii) add and subtract $n = 0$ terms.*

(c) Use the definition of a physical state (5.13) to comment on the implication of (5.15) for the spectrum of the bosonic string. *(1 point)*