Physikalisches Institut
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Theoretische Physik

Exercise 7
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Exercises Superstring Theory

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The ghost system

In the lecture you considered the Faddev-Popov quantization method by fixing a gauge worldsheet metric $\hat{h}_{\alpha\beta}$. The resulting partition function reads

$$Z = \int \mathcal{D}X^\mu \mathcal{D}h e^{iS[X,h]} \longrightarrow Z = \int \mathcal{D}X^\mu \mathcal{D}b\mathcal{D}c e^{iS[X,\hat{h},gh]}\,, \tag{7.1}$$

where $b_{\alpha\beta}$ and c^α are the anti-commuting *ghost fields.* Moreover, the action is now given by

$$S[X,\hat{h},b,c] = S_P[X,\hat{h}] + S_{gh}[b,c,\hat{h}]\,, \quad S_{gh}[b,c,\hat{h}] := -\frac{i}{2\pi}\int d^2\sigma\sqrt{-\hat{h}}\hat{h}^{\alpha\beta}b_{\beta\gamma}\hat{\nabla}_\alpha c^\gamma\,, \tag{7.2}$$

where S_P is the Polyakov action and S_{gh} is the action of the ghost system. One clearly sees that it would have been inconsistent to simply set $h_{\alpha\beta} = \eta_{\alpha\beta}$ and drop the $\mathcal{D}[h]$ integration, as it would have neglected the ghost contribution. To appreciate the ghost contribution, one also notices that the total energy momentum tensor given by

$$T_{\alpha\beta} := T^X_{\alpha\beta} + T^{gh}_{\alpha\beta}\,, \tag{7.3}$$

now gets a contribution $T^{gh}_{\alpha\beta}$ from the ghost action, aside of the already known contribution $T^X_{\alpha\beta}$ due to the Polyakov action. Note that this modifies the central charge term in the Virasoro algebra obtained in **Exercise sheet 5**.

For convenience we might choose the conformal gauge metric as fixed gauge, i.e. $\hat{h}_{\alpha\beta} = \eta_{\alpha\beta}$. Then the ghost system, consisting of Grassmann odd fields, is quantized by the following canonical anti-commutation relations[1]

$$\{b_{++}(\sigma,\tau), c^+(\sigma',\tau)\} = 2\pi\delta(\sigma-\sigma')\,, \quad \{b_{--}(\sigma,\tau), c^-(\sigma',\tau)\} = 2\pi\delta(\sigma-\sigma')\,. \tag{7.4}$$

(a) From (7.2) obtain the energy momentum tensor of the ghost system $T^{gh}_{\alpha\beta}$. What would be the equations of motion of the ghost fields? *(1 Point)*
Hint: Make variations over the fields in (7.2). Express your results in light-cone coordinates.

For the closed string the solutions of the equations of motion, periodic in σ with period ℓ are

$$c^\pm(\sigma,\tau) = \frac{\ell}{2\pi}\sum_{n\in\mathbb{Z}} c^\pm_n e^{-\frac{2\pi}{\ell}in\sigma_\pm}\,, \quad b_{\pm\pm}(\sigma,\tau) = \left(\frac{2\pi}{\ell}\right)^2 \sum_{n\in\mathbb{Z}} b^\pm_n e^{-\frac{2\pi}{\ell}in\sigma_\pm}\,. \tag{7.5}$$

Here $b^+_n := \bar{b}_n\,, b^-_n := b_n\,, c^+_n := \bar{c}_n\,, c^-_n := c_n$.

[1] Do not confuse here the anticommutator $\{\cdot,\cdot\}$ with the Poisson Brackets of previous exercises.

(b) Using the anti-commutation relations in (7.4) derive the anti-commutations of the ghost field Fourier modes given by *(1 Point)*

$$\{b_m, c_n\} = \delta_{m+n}\,, \quad \{b_m, b_n\} = \{c_m, c_n\} = 0\,. \tag{7.6}$$

(c) Recall **Exercise sheets 2 & 3**. Obtain the Virasoro generators of the ghost system as the conserved Noether charges *(1 Point)*

$$L_m^{gh} = -\frac{\ell}{4\pi^2}\int_0^\ell d\sigma e^{-\frac{2\pi}{\ell}im\sigma_-}T_{--}^{gh}(\sigma_-)\,, \quad \overline{L}_m^{gh} = -\frac{\ell}{4\pi^2}\int_0^\ell d\sigma e^{-\frac{2\pi}{\ell}im\sigma_+}T_{++}^{gh}(\sigma_+)\,. \tag{7.7}$$

To fully promote Virasoro generators to quantum operators, we need to take into account normal ordering $\star\cdots\star$ of the Fourier modes. Your result above should read

$$\textbf{Classical: } L_m^{gh} = \sum_{n\in\mathbb{Z}}(m-n)b_{m+n}c_{-n} \mapsto \textbf{Quantum: } \hat{L}_m^{gh} = \sum_{n\in\mathbb{Z}}(m-n)\star b_{m+n}c_{-n}\star\,. \tag{7.8}$$

Here $\star b_m c_{-n}\star = -c_{-n}b_m$ and $\star b_{-m}c_n\star = b_{-m}c_n$ for $m,n>0$. From now on, we drop the hats.

(d) Verify that $[L_m^{gh}, L_n^{gh}] = (m-n)L_{m+n}^{gh}$ for $m+n\neq 0$. *(1.5 Point)*

Similar to **Exercise Sheet 5**, normal ordering effects appear when $m+n=0$. This means the *Ghost Virasoro algebra* is also a central extension of the classical Virasoro algebra. A central extension $\hat{\mathfrak{g}} \simeq \mathfrak{g}\oplus \mathrm{c}\mathbb{C}$ of a Lie algebra $\mathfrak{g}$ by c satisfies

- $[X,Y]_{\hat{\mathfrak{g}}} = [X,Y]_{\mathfrak{g}} + \mathrm{c}P(X,Y)\ , \quad X,Y\in\mathfrak{g}\ ,$
- $[X,\mathrm{c}]_{\hat{\mathfrak{g}}} = 0\ ,$
- $[\mathrm{c},\mathrm{c}]_{\hat{\mathfrak{g}}} = 0\,,$

i.e. c belongs to the center of $\hat{\mathfrak{g}}$. Here $P:\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ is bilinear and antisymmetric. Similar to the case of the Virasoro algebra of the bosonic system, we note that $\mathrm{c}_{gh}P(L_m^{gh}, L_n^{gh}) = A^{gh}(m)\delta_{m+n}\,.$

(d) Show that $A^{gh}(m) = \frac{1}{12}(-26m^3+2m)\,.$ *(2.5 Point)*

Let us now look into the combined matter-ghost system. The total Virasoro generators are now given by

$$L_m = L_m^X + L_m^{gh} + a\delta_m\,, \tag{7.9}$$

where the last term accounts for a normal ordering ambiguity in $L_0^X + L_0^{gh}$. Then the Virasoro algebra of the total system follows

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n}\,. \tag{7.10}$$

(e) A non-vanishing total $A(m)$ translates to an anomaly of the local Weyl transformations. Verify that this anomaly is absent if and only if $d=26$ and $a=-1$. *(1 Point)*

The conformal group in d dimensions

Let M,N be smooth d-dimensional manifolds with metrics g and h respectively. A local diffeomorphism of open sets $\phi\colon U\subset M\to\phi(U)\subset N$, is called *a local conformal transformation* if

$$\phi^* h = \Lambda\cdot g \tag{7.11}$$

where Λ is a smooth scale function $\Lambda: U\to\mathbb{R}_{>0}$. For simplicity, consider the Euclidean d-dimensional spacetime $\mathbb{R}^{d,0}$ with metric $\eta_{\mu\nu} = \mathrm{diag}(1,...,1)$, μ,$\nu = 1,...,d$. Then, a conformal transformation of coordinates leaves the metric tensor invariant up to a scale, i.e.,

$$\eta'_{\mu\nu}(x'^\mu) = \eta_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} = \Lambda(x^\mu)\eta_{\mu\nu}(x^\mu)\ . \tag{7.12}$$

It preserves angles between any two arbitrary vectors on spacetime.

In this exercise, you will familiarize yourself with the conformal group in d dimensions and its algebra. Notice that it contains the Poincaré group as a subgroup (when $\Lambda(x^\mu) = 1$).

(a) Show that the consequence of requiring that an infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x^\mu), \ \epsilon(x^\mu) \ll 1, \tag{7.13}$$

is conformal (i.e., that it satisfies (7.12)) leads to *(1 Point)*

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f \eta_{\mu\nu} \ , \tag{7.14}$$

where

$$f = \frac{2}{d}(\partial \cdot \epsilon), \qquad \partial \cdot \epsilon = \partial_\mu \epsilon^\mu \ . \tag{7.15}$$

(b) Take the partial derivative ∂_ρ of (7.14), permute the indices of this resulting equation to find two similar equations. Now take a convenient linear combination of these three equations to find *(0.5 Point)*

$$2\partial_\mu \partial_\nu \epsilon_\rho = \eta_{\nu\rho}\partial_\mu f + \eta_{\rho\mu}\partial_\nu f - \eta_{\mu\nu}\partial_\rho f \ . \tag{7.16}$$

(c) Contract (7.16) with $\eta^{\mu\nu}$ and take ∂_ν of the resulting expression. Moreover, take ∂^2 of (7.14). Combine these results to get *(0.5 Point)*

$$(2-d)\partial_\mu \partial_\nu f = \eta_{\mu\nu}\partial^2 f \ . \tag{7.17}$$

Contracting (7.17) further with $\eta^{\mu\nu}$ leads to

$$(d-1)\partial^2 f = 0 \ . \tag{7.18}$$

From (7.18), one clearly sees that, for $d = 1$, there is no constraint on the function f. This means that any transformation in one dimension is conformal[2]. The 2-dimensional case will be studied in the next exercise. Let us now focus on the case $d \geq 3$.

(d) Equations (7.17) and (7.18) imply that $\partial_\mu \partial_\nu f = 0$, i.e., f is a linear function in the coordinates x^μ, $f(x^\mu) = A + B_\mu x^\mu$. Explain why this condition on f implies that ϵ_μ can be written as

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \ , \quad c_{\mu\nu\rho} = c_{\mu\rho\nu} \ , \tag{7.19}$$

where conditions on the coefficients a_μ, $b_{\mu\nu}$ and $c_{\mu\nu\rho}$ will be determined below. *(0.5 Point)*

Since (7.14) – (7.16) hold for all x^μ, we can treat each power of x^μ in (7.19) separately.

(e) Show that: *(1.5 Point)*

(i) there are no constraints on the constant term a_μ;

(ii) substitution of the linear term of (7.19) in (7.14) implies

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b^\rho{}_\rho \eta_{\mu\nu} \ ; \tag{7.20}$$

(iii) substitution of the quadratic term of (7.19) in (7.16) implies

$$c_{\mu\nu\rho} = \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu \ , \qquad b_\mu := \frac{1}{d} c^\sigma{}_{\sigma\mu} \ . \tag{7.21}$$

[2] In fact the notion of angle does not even exist in one dimension.

The term a_μ gives rise to an infinitesimal translation.
Moreover, (7.20) implies that b_μ can be separated into the sum of an antisymmetric part and a pure trace part as follows

$$b_{\mu\nu} = m_{\mu\nu} + \alpha\eta_{\mu\nu} \ , \qquad m_{\mu\nu} = -m_{\nu\mu} \ . \tag{7.22}$$

The antisymmetric part gives rise to infinitesimal rotations whereas the pure trace part gives rise to an infinitesimal scale transformation.
The infinitesimal transformation associated to $c_{\mu\nu\rho}$ is given by

$$x'^\mu = x^\mu + 2(b \cdot x)x^\mu - b^\mu x^2 \tag{7.23}$$

and it receives the name of *special conformal transformation* (SCT).
To each infinitesimal transformation, one gets a finite one, from which the generators of the conformal group can be identified.

The table below summarizes the finite conformal transformations together with the corresponding generators of the conformal group (translations and rotations form the usual Poincaré group).

Transformations		**Generators**
Translation	$x'^\mu = x^\mu + a^\mu$	$P_\mu = -i\partial_\mu$
Rotation	$x'^\mu = M^\mu_\nu x^\nu$	$L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$
Dilation	$x'^\mu = \alpha x^\mu$	$D = -ix^\mu\partial_\mu$
SCT	$x'^\mu = \frac{x^\mu - b^\mu x^2}{1-2b\cdot x+b^2x^2}$	$K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)$

The generators of the conformal group obey the conformal algebra given below

$$\begin{aligned}
&[D, P_\mu] = iP_\mu \\
&[D, K_\mu] = -iK_\mu \\
&[K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \\
&[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \\
&[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\
&[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho})
\end{aligned} \tag{7.24}$$

(f) Check the first four relations of (7.24). Use $[x_\mu, P_\nu] = i\eta_{\mu\nu}$. *(2 Points)*

In order to put the conformal algebra above into a simpler form, we define the following generators

$$\begin{aligned}
J_{\mu\nu} &= L_{\mu\nu} \ , \qquad J_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu) \ , \\
J_{-1,0} &= D \ , \qquad J_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu) \ .
\end{aligned} \tag{7.25}$$

It is not hard to show that the generators above satisfy the algebra of $SO(d+1,1)$, i.e.,

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}) \ , \tag{7.26}$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, 1, ..., 1)$.
In the similar case of Minkowski spacetime $\mathbb{R}^{d-1,1}$, where $\eta_{ab} = \mathrm{diag}(-1, -1, 1, ..., 1)$, the commutation relations satisfy the algebra of $SO(d, 2)$.