Physikalisches Institut
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Theoretische Physik

Exercise Sheet 9
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Exercises Superstring Theory

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1 Primary fields of the free boson CFT

We consider the 2d CFT of a free boson given in terms of the action

$$S[\phi] = 2g \int dz d\bar{z} \partial_z \phi(z,\bar{z}) \partial_{\bar{z}} \phi(z,\bar{z}) \,. \tag{1}$$

One can show easily that the holomorphic energy momentum tensor $T(z)$ of the free boson CFT is given by

$$T(z) = -2\pi g :\partial\phi(z)\partial\phi(z): , \tag{2}$$

Here, $\partial\phi$ is the chiral primary field of conformal weight $(h,\bar{h}) = (1,0)$ with the OPE

$$\mathcal{R}\Big(\partial_z\phi(z,\bar{z})\partial_w\phi(w,\overline{w})\Big) \sim -\frac{1}{4\pi g}\frac{1}{(z-w)^2} \,. \tag{3}$$

Now we want to study the spectrum of primary fields of the free boson conformal field theory.

(a) Show that the normal ordered operators $V_\alpha(z,\bar{z}) =:e^{i\alpha\phi(z,\bar{z})}:$ are primary fields and determine their conformal weights h and $\bar{h}$. *(2 Points)*
Hint: Determine the OPE with the energy momentum tensor $T(z)$.

(b) Consider two operators A and B linear in the creation and annihilation operators $a^\dagger$ and a of the harmonic oscillators, i.e., $[a^\dagger, a] = \mathbf{1}$. Show that these operators obey the relation

$$:e^A::e^B:=:e^{A+B}:\, e^{\langle AB\rangle} \,, \tag{4}$$

with $\langle AB\rangle = \langle 0| AB |0\rangle$. *(3 Points)*
Hint: Use the Baker-Campbell-Hausdorff formula.

$$e^{-A}Be^{A} = B + [B,A] + \frac{1}{2!}[[B,A],A] + \frac{1}{3!}[[[B,A],A],A] + \cdots \,. \tag{5}$$

(c) As the free bosonic field can be seen as a collection of decocupled harmonic oscillators, argue that the two-point correlation functions is given by *(3 Points)*

$$\langle V_\alpha(z,\bar{z})V_\beta(w,\overline{w})\rangle = \begin{cases} |z-w|^{-\frac{\alpha^2}{2\pi g}} & \text{for } \alpha = -\beta \\ 0 & \text{else} \end{cases} \,. \tag{6}$$

Hint: Derive the leading term in the OPE $\mathcal{R}\Big(V_\alpha(z,\bar{z})V_\beta(w,\overline{w})\Big)$, and use that $V_0(z,\bar{z})$ is the identity operator.

2 Operator algebra of primary fields

For a given CFT, the OPEs among all its primaries (incluiding regular terms) form the so-called **operator algebra**. The knowledge of the operator algebra determines all correlators of the CFT, i.e. it "solves" the CFT. Given two chiral primaries ϕ_k, ϕ_ℓ, scale invariance determines the structure of the operator algebra

$$\phi_k(z,\bar{z})\phi_\ell(0,0) = \sum_{[\phi_s]}\sum_{\{\vec{k}\}}\sum_{\{\vec{\bar{k}}\}} C^s_{k\ell}\beta^{s\{\vec{k}\}}_{k\ell}\beta^{s\{\vec{\bar{k}}\}}_{k\ell} z^{h_s-h_k-h_\ell+|\vec{k}|}\bar{z}^{\bar{h}_s-\bar{h}_k-\bar{h}_\ell+|\vec{\bar{k}}|}\phi_s^{\{\vec{k}\}\{\vec{\bar{k}}\}}(0,0)\,, \tag{7}$$

where $|\vec{k}| = \sum_n k_n$ and $\vec{k} = (k_1, k_2, \ldots k_m)\,, k_1 \leq k_2 \leq \cdots \leq k_m$. Similar for $\vec{\bar{k}}$. Here $C^s_{k\ell}$ are the structure constants determined by the three-point correlator of three pramaries $\langle\phi_s|\,\phi_k(z,\bar{z})\,|\phi_\ell\rangle$.

For simplicity in the following we assume two chiral primaries $\phi_k(z)$ and $\phi_\ell(z)$, with $h = h_k = h_\ell$, then

$$\phi_k(z)\phi_\ell(0) = \sum_s C^s_{k\ell} z^{h_s-2h}\psi_s(z)\,, \tag{8}$$

where

$$\psi_s(z) := \sum_{N=0}^{\infty}\sum_{\substack{\{\vec{k}\}\\ |\vec{k}|=N}} z^N\beta^{s\{\vec{k}\}}_{k\ell}L_{-\{\vec{k}\}}\phi_s(0)\,. \tag{9}$$

Hence

$$\psi_s(z)\,|0\rangle = \sum_{N=0}^{\infty} z^N\,|N;h_s\rangle\ , \tag{10}$$

where $|N;h_s\rangle$ is the level N state descending from $|\phi_s\rangle = |h_s\rangle$.

(a) Compute and obtain the following *(1 Point)*

$$L_n\phi_k(z)\phi_\ell(0)\,|0\rangle = \left[z^{n+1}\partial_z + (n+1)hz^n\right]\phi_k(z)\,|h_\ell\rangle\ . \tag{11}$$

(b) Moreover by acting $L_n\psi_s(z)\,|0\rangle$, derive the following relation *(1.5 Points)*

$$L_n\,|N+n;h_s\rangle = \left[h_s + (n-1)h + N\right]|N;h_s\rangle\ . \tag{12}$$

For low N, we can then begin to examine what descendant states are produced:

(c) **Level 1:** There is one descendant state *(1 Point)*

$$|1;h_s\rangle = \beta^{s\{1\}}_{k\ell}L_{-1}\,|h_s\rangle\ . \tag{13}$$

Using the relation given in (12), together with the Virasoro algebra, show that $\beta^{s\{1\}}_{k\ell} = \frac{1}{2}$.

(c) **Level 2:** There are two descendant states *(2.5 Point)*

$$|2;h_s\rangle = \beta^{s\{2\}}_{k\ell}L_{-2}\,|h_s\rangle + \beta^{s\{1,1\}}_{k\ell}L_{-1}L_{-1}\,|h_s\rangle\ . \tag{14}$$

Similarly as the previous exercise, using (12), find a pair of linearly independent equations for $\beta^{s\{1,1\}}_{k\ell}$ and $\beta^{s\{2\}}_{k\ell}$. The solution of such equations should read

$$\beta^{s\{1,1\}}_{k\ell} = \frac{\mathrm{c} - 12h - 4h_s + \mathrm{c}h_s + 8h_s^2}{4(\mathrm{c} - 10h_s + 2\mathrm{c}h_s + 16h_s^2)}\,,\quad \beta^{s\{2\}}_{k\ell} = \frac{2h - h_s + 4hh_s + h_s^2}{\mathrm{c} - 10h_s + 2\mathrm{c}h_s + 16h_s^2}\,. \tag{15}$$

Hint: You might need to evaluate the commutators $[L_1, L_{-1}^2]\,, [L_2, L_{-1}^2]\,, [L_1, L_{-2}]\,,$ *and* $[L_2, L_{-2}]\,.$