Exercises Superstring Theory

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1 Two-point correlation functions & Ward identities

A field $\phi(z, \bar{z})$ is called a **primary field** of conformal weight[1] $(h, \bar{h})$ if, under conformal transformations, $z \to z' = f(z)$, $\bar{z} \to \bar{z}' = f(\bar{z})$, it transforms as

$$\phi(z, \bar{z}) \to \phi'(z', \bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) \ . \tag{1}$$

A field which is not primary is called **secondary field**. Let $\tilde{\phi}$ be a secondary field, if (1) holds for $\tilde{\phi}$ only under global conformal transformations, $z', \bar{z}' \in SL(2, \mathbb{C})/\mathbb{Z}_2$, then $\tilde{\phi}$ is called a **quasi-primary field**[2]. Note that a primary field is always quasi-primary, but the reverse in general is not true.

Restricting only to the holomorphic part (also called chiral), a primary field transforms, under infinitesimal conformal transformations $z \to z + \epsilon(z)$, $\epsilon(z) \ll 1$, as

$$\phi(z) \to \phi'(z) = \phi(z) + \delta_\epsilon \phi(z) \qquad \text{with} \qquad \delta_\epsilon \phi(z) = -[\epsilon(z)\partial_z + h\partial_z \epsilon(z)]\phi(z) \ . \tag{2}$$

Consider the two-point correlation function of two quasi-primary fields $\phi_i = \phi_i(z_i)$ with conformal dimension h_i, $i = 1, 2$, in 2d CFT: $G(z_1, z_2) = \langle \phi_1(z_1)\phi_2(z_2) \rangle$.

(a) Show that conformal invariance of $G(z_1, z_2)$ implies

$$[\epsilon(z_1)\partial_{z_1} + h_1 \partial_{z_1}\epsilon(z_1) + \epsilon(z_2)\partial_{z_2} + h_2\partial_{z_2}\epsilon(z_2)]G(z_1, z_2) = 0 \ . \tag{3}$$

Hint: Recall $\delta_\epsilon \phi(z)$ from (2) and see what happens for $\delta_{\epsilon(z_1),\epsilon(z_2)}$ on $G(z_1, z_2)$. *(1 Point)*

For infinitesimal global conformal transformations one can show that $\epsilon(z) = \alpha + \beta z + \gamma z^2$ at first order in $\epsilon(z)$, where α, β and γ are constant infinitesimal parameters.

(b) Use $\epsilon(z_i) = \alpha$ in (3) to show that $G(z_1, z_2)$ depends on $x = z_1 - z_2$ only. *(1 Point)*

(c) Use $\epsilon(z_i) = \beta z_i$ in (3) to show that $G(z_1, z_2)$ is written as

$$G(z_1, z_2) = \frac{C_{12}}{(z_1 - z_2)^{h_1 + h_2}} \ . \tag{4}$$

where C_{12} is a constant. *(1 Point)*

[1] Also called dimension or scaling dimension.
[2] e.g., in the lecture you have seen that the energy-momentum tensor is a quasi-primary field.

(d) Use $\epsilon(z_i) = \gamma z_i^2$ in (3) to show that $G(z_1, z_2)$ vanishes unless $h_1 = h_2$. *(1 Point)*

In other words, two-point functions are given by

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = \begin{cases} \frac{C_{12}}{(z_1-z_2)^{h_1+h_2}} & h_1 = h_2, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

In the same way, one shows that three-point functions are given by

$$\langle \phi_1(z_1)\phi_2(z_2)\phi(z_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{13}^{h_1+h_3-h_2} z_{23}^{h_2+h_3-h_1}} \ , \tag{6}$$

where C_{123} is a constant and $z_{ij} = z_i - z_j$, $i = 1, 2, 3$.

In the following, we investigate Ward identities that encode invariance of correlation functions of quasi-primary fields under global conformal transformations. They are given by

$$\begin{aligned} &\sum_i \partial_{w_i} \langle \phi_1(w_1)...\phi_n(w_n)\rangle = 0 \ , \\ &\sum_i (w_i\partial_{w_i} + h_i)\langle \phi_1(w_1)...\phi_n(w_n)\rangle = 0 \ , \\ &\sum_i (w_i^2\partial_{w_i} + 2w_i h_i)\langle \phi_1(w_1)...\phi_n(w_n)\rangle = 0 \ . \end{aligned} \tag{7}$$

(e) Show explicitly that these identities are valid for the two- and three-point functions of (5) and (6). *(2.5 Points)*

2 Closed Bosonic String Theory as a CFT

In this exercise we make use of the learnt CFT techniques to work out a bit on closed bosonic string theory. Recall that the Polyakov action can be expressed as an action over the worldsheet complex coordinates, which is given by

$$S_P = \frac{1}{2\pi\alpha'}\int d^2z \partial X^\mu(z,\bar z)\bar\partial X_\mu(z,\bar z) \, . \tag{8}$$

The solutions of the $X(z, \bar z)$ fields can be derived from the equations of motion, which read

$$X(z,\bar z) = x_0^\mu - i\frac{\alpha'}{2}p^\mu \log|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}-\{0\}}\frac{1}{n}(\alpha_n z^{-n} + \bar\alpha_n \bar z^n) \, . \tag{9}$$

Moreover, you might recall that (9) can be seperated in terms of right and left moving coordinates of the bosonic strings given by $X^\mu(z)$ and $\overline{X}^\mu(\bar z)$ respectively, i.e.

$$X^\mu(z,\bar z) = X^\mu(z) + \overline{X}^\mu(\bar z) \, . \tag{10}$$

(a) Derive the two-point correlation functions *(1 Point)*

$$\langle \partial X^\mu(z)\partial X^\nu(w)\rangle = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-w)^2} \, , \quad \langle \bar\partial X^\mu(\bar z)\bar\partial X^\nu(\bar w)\rangle = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(\bar z-\bar w)^2} \, . \tag{11}$$

The energy momentum tensor following from (8) is given by

$$T(z) = -\frac{1}{\alpha'} : \partial X^\mu(z) \partial X_\mu(z) : , \tag{12}$$

and likewise for $\overline{T}(\bar{z})$.

(b) Compute explicitly the following OPE *(2 Points)*

$$T(z)T(w) = \frac{\frac{d}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{reg}. \tag{13}$$

Recall **Exercise sheet 4**, due to logarithmic dependence in the correlation functions of the bosonic fields $X(z)$ and $\overline{X}(\bar{z})$, such fields are not primary fields. However, their derivatives and the fields $V_k(z,\bar{z}) = e^{ik_\mu X^\mu(z,\bar{z})}$ are indeed primary fields. For the latter case, you proved in **Exercise sheet 9** its respective conformal weight: $(\frac{\alpha'}{4}k^2, \frac{\alpha'}{4}k^2)$.

(c) Show that $\partial X^\mu(z)$ is a primary field of conformal weight (1,0). Similarly holds for $\bar{\partial} X^\mu(\bar{z})$ with conformal weight (0,1). *(1 Point)*

(d) Higher derivatives $\partial^n X(z)$ are not primary but descendant fields with $h = n$. Prove that $\partial^2 X(z)$ has conformal weight $(2,0)$. *(1 Point)*

Physical string states correspond to primary fields satisfying the *physical state conditions* together with the *level matching conditions*, given by the asymptotic states

$$|\phi\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z}) |0\rangle , \tag{14}$$

where $\phi(z,\bar{z})$ are conformal fields referred to **vertex operators**. e.g, in the spectrum of the closed bosonic string the lowest possible state is the tachyon $|k\rangle$ given by the space-time scalar with momentum k

$$\textbf{Tachyon:} \quad |k\rangle = \lim_{z,\bar{z}\to 0} V_k(z,\bar{z}) |0\rangle . \tag{15}$$

(e) Verify that the physical state and level matching conditions require $V_k(z,\bar{z})$ having conformal weight (1,1). What does this imply for $|k\rangle$? *(1 Point)*

(f) Compute the following *(1.5 Point)*

$$\partial X^\mu(z) V_k(z,\bar{z}) = -\frac{\alpha'}{2} \frac{ik^\mu}{(z-w)} V_k(z,\bar{z}) + \text{reg}. \tag{16}$$

Moreover, verify that $p^\mu |k\rangle = k^\mu |k\rangle$.
Hint: Note that the momentum operator is given by $p^\mu = \sqrt{2}\alpha' \alpha_0^\mu = \frac{2}{\alpha'} \oint \frac{dz}{2\pi i} i\partial X^\mu$.

The next level states are in the spectrum are of the following form

$$\textbf{1st states:} \quad |k,\xi\rangle := -\frac{2}{\alpha'} \xi_{\mu\nu} \lim_{z,\bar{z}\to 0} : \partial X^\mu(z) \bar{\partial} X^\nu(\bar{z}) e^{ik_\mu X^\mu(z,\bar{z})} : |0\rangle = \xi_{\mu\nu} \alpha_{-1}^\mu \bar{\alpha}_{-1}^\nu |k\rangle . \tag{17}$$

Here $\xi_{\mu\nu}$ is a *polarization tensor*.

(g) Show that the OPE with the vertex operator of $|k,\xi\rangle$, i.e. $T(z) : \partial X^\mu(z) \bar{\partial} X^\nu(\bar{z}) e^{ik_\mu X^\mu(z,\bar{z})} :$ is that of a primary field whenever *(2 Points)*

$$k^\mu \xi_{\mu\nu} = \xi_{\mu\nu} k^\nu = 0 . \tag{18}$$

(h) What further conditions need to be imposed for $|k,\xi\rangle$ in order to achieve conformal weight $(1,1)$? What can you say about the cases when $\xi_{\mu\nu}$ is symmetric traceless, antisymmetric and pure trace? *(2 Points)*