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Exercise Sheet 11
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Exercises Superstring Theory

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1 String Scattering Amplitudes

In the context of the perturbative bosonic string theory defined on a given worldsheet, i.e. a Riemann surface Σ, an N-particle amplitude is essentially given by

$$A(\Lambda_1, k_1; \ldots ; \Lambda_N, k_N) = C_\Sigma \int \mathcal{D}h\mathcal{D}X \int d^2z_1 \cdots d^2z_N V_{\Lambda_1}(z_1, \bar{z}_1) \cdots V_{\Lambda_N}(z_n, \bar{z}_N) e^{-S_P[X,h]} \,. \quad (1)$$

Here Λ_i represents a sate of a given type with momentum k_i. For instance in **Exercise sheet 10**, for the closed bosonic string, we determined that **graviton** states $|G, k\rangle$ are generated by the Vertex operator V_{G_k}

$$|G, k\rangle = lim_{z,\bar{z}\to 0} V_{G_k}(z, \bar{z}) \,|0\rangle \;, \quad V_{G_k}(z, \bar{z}) = -\frac{2}{\alpha'}\zeta_{\mu\nu} : \partial X^\mu(z)\bar{\partial}X^\nu(\bar{z})e^{ik\cdot X(z,\bar{z})} : \,. \quad (2)$$

Here $\zeta_{\mu\nu}$ denotes the symmetric traceless polarization tensor. Essentially the amplitude in (1) can be computed in terms of the N-point correlation functions (defined on Σ) as follows

$$A(\Lambda_1, k_1; \ldots ; \Lambda_N, k_N) = \text{const} \times g_s^{N-2} \int \frac{d^2z_1 \cdots d^2z_N}{V_{CKG}} \langle V_{\Lambda_1}(z_1, \bar{z}_1) \cdots V_{\Lambda_N}(z_N, \bar{z}_N)\rangle_\Sigma \,. \quad (3)$$

Here g_s is the string coupling constant and V_{CKG} is the volume of the conformal Killing group of Σ. In the following, we discuss the simplest processes at tree-level for the bosonic string. This corresponds to amplitudes on Riemann surfaces with positive Euler characteristic. In particular, correlation functions on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \simeq S^2$ yield closed string amplitudes. On the other hand, the open string tree-level worldsheet has one boundary. Hence the latter may be mapped to the disk $D_2 = \{z \in \mathbb{C} : |z| \leq 1\}$ or to the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}z > 0\}$. The purpose of this Exercise sheet will be the amplitude for two-to-two scattering of tachyons in the closed bosonic string theory.

2 The Virsasoro-Shapiro amplitude

The simplest non-trivial amplitude of the closed bosonic string is given by the four-tachyon amplitude, which reads[1]

$$A_{VS}(k_1, k_2, k_3, k_4) = g_c^4 C_{\mathbb{P}^1} \int d^2z_4 \langle \prod_{j=1}^{3} : \bar{c}ce^{ik_j\cdot X}(z_j, \bar{z}_j) :: e^{ik_4\cdot X}(z_4, \bar{z}_4) :\rangle_{\mathbb{P}^1} \,. \quad (4)$$

Since the bosonic X-CFT and the ghost sector bc-CFT are independent, the problem factorizes in the computation of the following correlation functions:

[1] The presence of conformal Killing vectors is taken care of if we drop the integration over the positions of three of the vertex operators and multiply each of them by $\bar{c}(\bar{z}_i)c(z_i)$.

(a) Recall the raw version of Wick's theorem

$$:e^{A_1}:\cdots:e^{A_N}:=\prod_{i<j}e^{\langle A_iA_j\rangle}:e^{A_1+\cdots+A_N}:\,. \tag{5}$$

Using (5) obtain the following result *(1 Point)*

$$\langle :e^{ik_1\cdot X(z_1,\bar z_1)}:\cdots:e^{ik_N\cdot X(z_N,\bar z_N)}:\rangle_{\mathbb{P}^1}=\text{const}\cdot\delta\Big(\sum_{i=1}^N k_i\Big)\prod_{j<l}|z_j-z_l|^{\alpha' k_j\cdot k_l}\,. \tag{6}$$

Note that A_i in (5) is an operator linear in annihilation and creator operators. The factor $\delta(\sum k_i)$ in (6) appears due to the zero modes terms in $X^\mu(z_i,\bar z_i)$.

(b) For the ghost sector 3-point correlation function derive the following relation *(1 Point)*

$$\langle\prod_{i=1}^3:\bar c c(z_i,\bar z_i):\rangle_{\mathbb{P}^1}=\text{const}\cdot|z_{12}z_{23}z_{13}|^2\,,\quad z_{ij}:=z_i-z_j\,. \tag{7}$$

*Hint: You might consider the form of the 3-point function given in **Exercise sheet 10**. Take as a fact that c ($\bar c$) is a fermionic holomorphic (antiholomorphic) field with conformal weight $(h,\bar h)=-(1,0)$ $\Big((h,\bar h=(0,-1)\Big)$.*

By $PSL(2,\mathbb{C})$ invariance of the final amplitude we can fix z_1,z_2,z_3 to convenient positions. We consider $z_1=0,z_2=1,z_3=\infty$. For convenience, rename $z:=z_4$.

(c) After fixing the above mentioned points, collect factors and rewrite $A_{VS}(k_1,k_2,k_3,k_4)$ in terms of the following non-trivial integral *(1 Point)*

$$I:=\int d^2z|z|^{-\alpha' k_1\cdot k_2}|1-z|^{\alpha' k_2\cdot k_4}\,. \tag{8}$$

Now we introduce the well known **Mandelstam variables**

$$s=-(k_1+k_2)^2\,,\quad t=-(k_1+k_3)^2\,,\quad u=-(k_1+k_4)^2\,. \tag{9}$$

(d) Verify the following equation $s+t+u=-\frac{16}{\alpha'}$. Moreover, show that (8) can be expressed as *(2 Points)*

$$I=\int d^2z|z|^{-\alpha'\frac{u}{2}-4}|1-z|^{\alpha'\frac{t}{2}-4}\equiv J(s,t,u)\,. \tag{10}$$

It turns out that the function $J(s,t,u)$ can be rewritten in terms of the **Euler Γ-functions**. Without proof, we now quote the following integral

$$C(a,b):=\int d^2z|z|^{2a-2}|1-z|^{2b-2}=2\pi\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)}\,,\quad a+b+c=1\,. \tag{11}$$

(e) Making the right identification for a and b in (11), arrive at the well known expression of the **Virasoro-Shapiro amplitude** given by *(2 Points)*

$$A_{VS}(k_1,k_2,k_3,k_4)=ig_c^2C_{\mathbb{P}^1}(2\pi)^d\delta\Big(\sum_i k_i\Big)\frac{\Gamma(-1-\frac{\alpha'}{4}s)\Gamma(-1-\frac{\alpha'}{4}t)\Gamma(-1-\frac{\alpha'}{4}u)}{\Gamma(2+\frac{\alpha'}{4}s)\Gamma(2+\frac{\alpha'}{4}t)\Gamma(2+\frac{\alpha'}{4}u)}\,. \tag{12}$$

It is easy to verify that A_{VS} is symmetric in t and s (and u). In the following consider a fixed Mandelstam variable t with s varying.

(f) Where are the poles of $\Gamma(-1 - \frac{\alpha'}{4}s)$ located? What can you associate with such s values in the closed bosonic string theory? *(2 Points)*

(g) Consider the scattering process $1 + 2 \to 3 + 4$ in the kinematic limit corresponding to scattering at high energies and fixed angle θ, with $s \to \infty, t \to \infty$ and $\frac{s}{t}$ fixed. Using the asymptotic behavior $\Gamma \to e^{x \ln x}$ for $Re(x) \to \infty$, deduce *(2 Points)*

$$A_{VS}(k_1, k_2, k_3, k_4) \to \exp\Big(- \frac{\alpha'}{2}(s \ln s + t \ln t + u \ln u)\Big), \text{ for } |s|, |t| \to \infty \text{ and } \frac{s}{t} \text{ fixed.} \quad (13)$$

Write a short sentence about the physical implication of this result.

The computations performed in this section (tree level) can be carried out for other states in the spectrum. This leads to similar results to those observed for the scattering of tachyons. Of special interest would be the scattering process of gravitons[2] $G_{k_1} + G_{k_2} \to G_{k_3} + G_{k_4}$. The interactions can be compared with the vertices from a in the 26-dimensional ambien spacetime. However, the amplitudes corresponding to the latter (26d Hilbert-Einstein action) diverge. On the other hand, if we restrict attention to low-energies, one can show that these coincide with the amplitudes of gravitons of the closed string theory. This is the first place to see that UV problems of general relativity might have a good chance of being cured in string theory.

[2] note that in this case the computation of $\langle V_{G_{k_1}}(z_1, \bar{z}_1) V_{G_{k_2}}(z_2, \bar{z}_2) V_{G_{k_3}}(z_3, \bar{z}_3) V_{G_{k_4}}(z_1, \bar{z}_4) \rangle_{\mathbb{P}^1}$ would be much more involved.