Physikalisches Institut
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Theoretische Physik

Exercise 00
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Exercises on General Relativity and Cosmology

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–Class Exercises–

Exercise 0.1: Lorentz boosts

Consider a boost in the x-direction with speed $v_A = \tanh(\alpha)$ followed by a boost in the y-direction with a speed $v_B = \tanh(\beta)$. Show that the resulting Lorentz transformation is the same as doing a pure rotation followed by a pure boost, and determine the rotation and boost.

Exercise 0.2: Electromagnetism

Maxwell's equations are given by

$$\overrightarrow{\nabla} \cdot \overrightarrow{E} = \epsilon, \quad \overrightarrow{\nabla} \times \overrightarrow{B} = \frac{\partial \overrightarrow{E}}{\partial t} + \overrightarrow{J}, \quad \overrightarrow{\nabla} \cdot \overrightarrow{B} = 0, \quad \overrightarrow{\nabla} \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t}$$

To make their properties under Lorentz transformation explicit, we can choose an anti-symmetric 4×4 tensor, $F^{\mu\nu}$, the *electromagnetic field tensor* and the (charge and current) source density four-vector, J^μ, as:

$$F^{\mu\nu}(t, \mathbf{x}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$J^\mu(t, \mathbf{x}) = (\epsilon,\ \mathbf{J}) = \left(e\delta^3(\mathbf{x} - \mathbf{x}(t)),\ e\delta^3(\mathbf{x} - \mathbf{x}(t)) \frac{d\mathbf{x}(t)}{dt} \right)$$

(a) Show that $\partial_\mu F^{\mu\nu} = -J^\nu$ and $\epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0$ reproduce Maxwell's equations. What are the Lorentz invariants that can be constructed?

(b) Verify in the rest frame that

$$f^\mu \equiv \frac{dp^\mu}{d\tau} = eF^\mu{}_\nu \frac{dx^\nu}{d\tau}$$

is the correct equation for the electromagnetic four-force f^μ acting on a charged particle. ($p^\mu = m\ dx^\mu / d\tau$)

Exercise 0.3: Energy-momentum tensor

In analogy to the electrical charge and current density vector above, we can define a "charge" and "current density" for the matter 4-momentum, p^μ, the *matter energy-momentum tensor*:

$$T_m^{\mu\nu}(\mathbf{x}, t) \equiv p^\mu(t)\frac{dx^\nu(t)}{dt}\delta^3(x - x(t))$$

(a) Show that the energy-momentum tensor is only conserved up to a *force density*, G^μ which vanishes for *free* particles:

$$\partial_\nu T^{\mu\nu} = G^\mu$$

(b) Check that for the (interacting) electromagnetic quantities given above, we get G^μ to be $F^\mu{}_\nu J^\nu$.

(c) The *electromagnetic energy-momentum tensor* was defined in the lecture as:

$$T_{em}^{\mu\nu} \equiv F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

Show that the divergence of this cancels with that of G^μ defined in (b), so that $T_m^{\mu\nu} + T_{em}^{\mu\nu}$ is conserved.