Physikalisches Institut
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Theoretische Physik

Exercise 03
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Exercises on General Relativity and Cosmology

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–Home Exercises–
Due 4 May 2011

Exercise 3.1: Lagrange formalism with generalized coordinates ***(7 credits)***

(a) Obtain the equations of motion for the general coordinates q^k with metric $g_{ij}(q)$ from the Lagrange equation:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^k} - \frac{\partial \mathcal{L}}{\partial q^k} = 0$$

where $\mathcal{L} = T(\dot{q}, q) - V(q)$. Take the KE to be

$$T = \frac{m}{2} g_{ij}(q)\ \dot{q}^i \dot{q}^j$$

(3 credits)

(b) Verify that the e.o.m in (a) take the form:

$$\ddot{q}^l + \Gamma^l{}_{ij}\dot{q}^i\dot{q}^j = -\frac{1}{m} g^{lk}\frac{\partial V}{\partial q^k} \tag{1}$$

(2 credits)

(c) Show that the Christoffel symbols above, $\Gamma^l{}_{ij}$, are of the usual form:

$$\Gamma^l{}_{ij} = \frac{1}{2} g^{lk}\left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k}\right)$$

(2 credits)

Remark: Note that the variational principle, $\delta \int (g_{ij}\dot{x}^a\dot{x}^b) = 0$ gives the same geodesics as the defining property for geodesics, $\delta \int ((g_{ij}\dot{x}^a\dot{x}^b))^{\frac{1}{2}} = 0$, where s is any affine parameter like, for eg, the proper length. This variation gives the Christoffel connection (torsionless and metric-compatible), irrespective of any other connection that may be defined on the manifold. So, in practice, a very fast way of computing Christoffel symbols is to write down the Euler-Lagrange equations for the simplified action and then read off the Christoffel symbols from the resulting geodesic equation.

Exercise 3.2: Christoffel symbols for rotating coordinates ***(7 credits)***

Fictitious forces that are considered in non-inertial frames in Newtonian mechanics can be seen to arise from the geometry that describes the frame. As an example, let us consider the rotating coordinates system from exercise 1.2(b) :

$$t' = t\ ;\quad x' = (x^2+y^2)^{\frac{1}{2}}\ cos(\phi - \omega t)\ ;\quad y' = (x^2+y^2)^{\frac{1}{2}}\ sin(\phi - \omega t);\quad z = z';\quad tan(\phi) = y/x$$

(a) Use the results of exercise(3.1) to calculate the e.o.m for a particle with a flat potential in the non-inertial rotating coordinates given above (*4 credits*)

(b) Rearrange the result and identify the terms that describe the centrifugal and coriolis forces (both fictitious) that arise in a rotating frame (*3 credits*)

Exercise 3.3: Christoffel symbols for a diagonal metric ***(6 credits)***

For a diagonal metric, prove that the Christoffel symbols are given by

(a) $\Gamma^{\mu}{}_{\nu\lambda} = 0$ (*1 credit*)

(b) $\Gamma^{\mu}{}_{\lambda\lambda} = -\dfrac{1}{2g_{\mu\mu}}\dfrac{\partial g_{\lambda\lambda}}{\partial x^{\mu}}$ (*2 credits*)

(c) $\Gamma^{\mu}{}_{\mu\lambda} = \dfrac{\partial}{\partial x^{\lambda}}\left(log(|g_{\mu\mu}|^{\frac{1}{2}}\right)$ (*1 credit*)

(d) $\Gamma^{\mu}{}_{\mu\mu} = \dfrac{\partial}{\partial x^{\mu}}\left(log(|g_{\mu\mu}|^{\frac{1}{2}}\right)$ (*2 credits*)

Here, $\mu \neq \nu \neq \lambda \neq \mu$ and there is no summation over repeated indices.