Physikalisches Institut
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Theoretische Physik

Exercise 04
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Exercises on General Relativity and Cosmology

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Due 11 May 2011

Exercise 4.1: Lie derivative ***(7 credits)***

The Lie derivative (or directional derivative) is yet another method to determine how a (p,q)-tensor is changing over nearby points. It is different from the covariant derivative. The covariant derivative relies on the specification of an affine connection, which explains locally how the choice of a basis set for vectors changes from point to point over a manifold, while the Lie derivative relies on the behavior of a vector field as defining a congruence of curves (flows) defined in the neighborhood of the point. The definition of the Lie derivative of a geometric object $\Phi^A[x^\mu(P)]$ (where, A represents *all* the tensor indices, $x^\mu(P)$ are the coordinates of a point P) is as follows:

Make an infinitesimal point transformation $P_0 \to P_N$ by $x^\mu(P_0) = x^\mu(P_N) + \xi^\mu(P_N)$. Also make an infinitesimal coordinate transformation that makes the numerical values of the coordinates of P_N the same as those of P_0 in the original coordinates: $\overline{x}^\mu(P_N) = x^\mu(P_0)$. Then the Lie derivative is:

$$\mathcal{L}_\xi \Phi^A(P_0) = \lim_{\xi\to 0}[\Phi^A(P_0) - \overline{\Phi}^A(P_N)] \tag{1}$$

(a) Show that the Lie derivative for a vector field Y, obtained using the above definition, is same as the one from the following definition:

$$\mathcal{L}_\xi Y \equiv [\xi, Y] = \xi^\beta Y_{,\beta} - Y^\beta \xi_{,\beta}$$

where, semi-colon (;) implies a covariant derivative and comma (,) implies a usual derivative. *(3 credits)*

(b) Using definition (1), show that the Lie derivative of a (1,1) tensor, $Y^\mu{}_\nu$, is given by:

$$\mathcal{L}_\xi Y^\mu{}_\nu = \xi^\eta Y^\mu{}_{\nu,\eta} - \xi^\mu{}_{,\eta} Y^\eta{}_\nu + \xi^\eta{}_{,\nu} Y^\mu{}_\eta$$

Remark: Connection coefficients cancel out from the expression for the Lie derivatives even if commas are replaced by semicolons *(4 credits)*

Exercise 4.2: Killing vector field ***(9 credits)***

The Lie derivative of a (0,2) metric tensor is given by:

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\eta g_{\mu\nu;\eta} + \xi^\eta{}_{;\mu} g_{\eta\nu} + \xi^\eta{}_{;\nu} g_{\mu\eta} = \xi_{\nu;\mu} + \xi_{\mu;\nu}$$

where, metric-compatibility of the connection (ie. $g_{\mu\nu;\eta} = 0$) is used in the second reduction. A Killing vector field is one which preserves the metric along its direction ie. $\mathcal{L}_\xi g_{\mu\nu} = 0$ (Killing equation).

(a) Solve the Killing equation to find the Killing vector fields of the 2-sphere:

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

Show that the resulting general Killing vector is a linear combination of three (Killing) vectors, each of which can be identified as the usual angular momentum operators along the three axis of the ambient (embedding) space. (*4 credits*)

(b) In flat Minkowski spacetime, find ten Killing vectors that are linearly independent.
Remark: These are the translation, rotation and boost vectors (*3 credits*)

(c) If ξ is a Killing vector and T is the stress-energy tensor, show that $J^\mu \equiv T^{\mu\nu}\xi_\nu$ is a conserved quantity ie. $J^\mu{}_{;\mu} = 0$.
Hint: You may use $T^{\mu\nu}{}_{;\mu} = 0$.
Remark: For the Minkowski case, the corresponding conserved Noether charges are the 4 linear momentum and the 6 spacetime angular momentum of rotation and boost (*2 credits*)

Exercise 4.3: Divergence of a vector ***(4 credits)***

Prove the following identity for the divergence (on a manifold with Christoffel connection) of a vector field:

$$J^\alpha{}_{;\alpha} = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} J^\alpha \right)_{,\alpha}$$

where, $g = \det(g_{\mu\nu})$.
Hint: For the derivative of the determinant, you may use the following:

$$\left(\ln|\det A|\right)_\alpha = \mathrm{tr}\left(A^{-1}A_{,\alpha}\right)$$

Remark: The identity can be extended to any *antisymmetric* (p,0) tensor as:

$$F^{\alpha\beta\ldots}{}_{;\alpha} = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} F^{\alpha\beta\ldots} \right)_{,\alpha}$$