Physikalisches Institut
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Theoretische Physik

Exercise 05
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# Exercises on General Relativity and Cosmology

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–Home Exercises–
Due 18 May 2011

**Exercise 5.1: Riemann normal coordinates** ***(7 credits)***

As was shown in class, Riemann normal coordinate system, at some point on the manifold, is formed from the tangents to the geodesics through that point such that the Christoffel symbols vanish locally.

(a) Argue why the metric can be locally written as a Minkowski flat metric in these coordinates. (*1 credit*)

(b) Starting from the definition of covariant derivative of vector fields, derive the expression for the covariant derivative of $g_{\mu\nu}$ (*4 credits*)

(c) Assuming a metric-compatible connection (ie. covariantly constant metric), derive the value of $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}$. Justify the validity of the statement that these coordinates are locally inertial. (here, ˆ denotes the Riemann normal coordinates) (*2 credits*)

Remark: Because of these coordinates, there exists a local *exponential map* from the tangent space to the manifold, $\mathsf{exp}_p : T_pM \to M$ defined in exactly the same way as the exponential map from a Lie algebra to its Lie group. The range and the domain of the exponential map may not include all of $M$ and $T_pM$ respectively. The latter cases are termed *geodesically incomplete* and those points $p$ are *singularities* of the manifold. This is often the working definition of singularities, as used in General Relativity (like in the case of black holes and big bang).

**Exercise 5.2: Parametrization of geodesics** ***(7 credits)***

The usual definition of geodesics can be expressed in the form ($T^\alpha$ is tangent to the curve ie. $T^\alpha = dx^\alpha/d\lambda$):

$$T^\alpha \nabla_\alpha T^\beta = 0 \tag{1}$$

1. Show that eq(1) reproduces the usual form of the geodesic:

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0$$

(*2 credits*)

2. An affine parameter $\lambda$ is one for which the equation of geodesic motion takes the above form. Show that all affine parameters are related by linear transformations with constant coefficients. *(2 credits)*

3. Intuitively, geodesics are "straightest possible lines" one can draw in a curved geometry. To satisfy this intuitive requirement, one might require only that the tangent vector to the geodesic curve point in the same direction as itself when parallel propagated, and not demand that it maintain the same length. This would yield the weaker condition: $T^\alpha \nabla_\alpha T^\beta = \alpha T^\beta$ (where, $\alpha$ is any arbitrary function on the curve). Show that any curve whose tangent satisfies this weaker condition can be reparametrized so that eq (1) is also satisfied. Thus, there is no true loss of generality in neglecting the weaker definition. *(3 credits)*

**Exercise 5.3: Hodge duality** ***(4 credits)***

Let $\alpha$ be a p-form on an $n$-dimensional oriented manifold with metric $g_{ab}$, ie. $\alpha_{a_1 \dots a_p}$ is a totally antisymmetric tensor field. We define the Hodge dual, $*\alpha$, of $\alpha$ by:

$$*\alpha_{b_1 \dots b_{n-p}} = \frac{1}{p!}\sqrt{|g|}\alpha^{a_1 \dots a_p}\epsilon_{a_1 \dots a_p b_1 \dots b_{n-p}}$$

where $\epsilon$ is the Levi-Civta symbol. Show that:

$$* * \alpha = (-1)^{s+p(n-p)}\alpha$$

where $s$ is the number of minuses occuring in the signature of $g_{ab}$. *(4 credits)*