Physikalisches Institut
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Theoretische Physik

Exercise 06
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# Exercises on General Relativity and Cosmology

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–Home Exercises–
Due 25 May 2011

**Exercise 6.1: Vielbein** ***(15 credits)***

If $g = g_{\mu\nu}(x)\ dx^\mu \otimes dx^\nu$ is the usual metric (0,2)-tensor, *at each point*, we can always introduce a set of abstract orthonormal basis vectors, $\hat{e}_{(a)}$, called vielbein (or **tetrads** or **non-coordinate bases**) such that $g(\hat{e}_{(a)}, \hat{e}_{(b)}) = \eta_{ab}$ at that point (note: In a Euclidean spacetime, $\eta$ is the usual Euclidean metric. In our cases below, it is Minkowski). At each point, we can express our old coordinate basis, $\partial_\mu \equiv \hat{e}_{(\mu)}$, in the new basis as: $\hat{e}_{(\mu)} = e_\mu{}^a\ \hat{e}_{(a)}$, so that the above orthonormality condition becomes:

$$g_{\mu\nu}(x)\ e^\mu{}_a(x)\ e^\nu{}_b(x) = \eta_{ab}(x)$$

where, $e^\mu{}_a$ and $e_\mu{}^a$ are related by raising and lowering of indices with $g_{\mu\nu}$ and $\eta_{ab}$. Also, $e_\nu{}^a$ can be seen as the components of a (1,1)-tensor of mixed bases: $e_\nu{}^a\ dx^\nu \otimes \hat{e}_{(a)}$.

(a) Justify why we can demand tensors with vielbein indices to be covariant under *local* Lorentz transformations of the vielbein bases. (*1 credit*)

(b) Find a relation between vielbein, its spin connection coefficients and Christoffel connection coefficients. (*4 credits*)
Hint: Take a vector in coordinate basis, $X = X^\mu \partial_\mu$, and write down its covariant derivative (a (1,1)-tensor) in terms of Christoffel connection coefficients. Next, write down the same vector in terms of vielbein, $X = X^a\ \hat{e}_{(a)}$, and its covariant derivative (a (1,1)-tensor with mixed basis ie. $dx^\nu \otimes \hat{e}_{(a)}$) in terms of spin connection coefficients introduced in class. Transform the latter (mixed) basis into the former (pure coordinate) basis and compare the coefficients to get the result.

(c) Rearrange the above result to show that the covariant derivative of vielbein vanishes ie. $\nabla_\mu e_\nu{}^a = 0$ (*3 credits*)
Remark: Note that this result holds in general and does not require the connection to be torsion-free or metric-compatible.

(d) Derive the transformation of spin connection under local Lorentz transformation (as given in the lecture):

$$\omega_\mu{}^{a'}{}_{b'} = \Lambda^{a'}{}_a \Lambda^b{}_{b'} \omega_\mu{}^a{}_b - \Lambda^c{}_{b'} \partial_\mu \Lambda^{a'}{}_c$$

(*3 credits*)

(e) $X_\mu{}^a$ can be viewed as a vector-valued 1-form. Its **exterior derivative**, $(dX)_{\mu\nu}{}^a \equiv \partial_\mu X_\nu{}^a - \partial_\nu X_\mu{}^a$, transforms covariantly as a (0,2)-tensor under General Coordinate Transformation (for indices $\mu, \nu$), but not as a (1,0)-tensor under Local Lorentz Transformation (for index $a$). Show that this problem is rectified if the expression is modified as:

$$(dX)_{\mu\nu}{}^a + (\omega \wedge X)_{\mu\nu}{}^a \equiv \partial_\mu X_\nu{}^a - \partial_\nu X_\mu{}^a + \omega_\mu{}^a{}_b X_\nu{}^b - \omega_\nu{}^a{}_b X_\nu{}^b$$

which transforms covariantly in all its indices. (*4 credits*)

Remark: As was remarked in class, vielbein provides an abstract frame to handle spinor fields, and hence, similarities between (general coordinate transformations in) GR and (local gauge transformations in) gauge theories are easier to see in this formalism.

**Exercise 6.2: Holonomy** ***(5 credits)***

It turns out to be possible to write down a general solution to the equation of parallel transport,

$$\frac{dx^\mu}{d\lambda}\partial_\mu V^\nu + \frac{dx^\mu}{d\lambda}\Gamma^\nu{}_{\mu\sigma}V^\sigma = 0.$$

We begin by noticing that for some path $\gamma$ connecting the points $\lambda$ and $\lambda_0$, solving the above equation amounts to finding a matrix $P^\mu{}_\rho(\lambda, \lambda_0)$, called the **parallel propagator**, such that, $V^\mu(\lambda) = P^\mu{}_\rho(\lambda, \lambda_0)V^\rho(\lambda_0)$. Clearly, $P^\mu{}_\rho$ depends on the path $\gamma$.

1. Derive the following (where, $A^\mu{}_\rho = -\Gamma^\mu{}_{\sigma\rho}\frac{dx^\sigma}{d\lambda}$):

$$P^\mu{}_\rho(\lambda, \lambda_0) = \delta^\mu_\rho + \int_{\lambda_0}^{\lambda} A^\mu{}_\sigma(\eta)P^\sigma{}_\rho(\eta, \lambda_0)$$

(*2 credits*)

2. Show that solving the above by iterating along the path (ie. taking the right-hand side and plugging it into itself repeatedly), one arrives at the following:

$$P^\mu{}_\rho(\lambda, \lambda_0) = \delta^\mu_\rho + \sum_{n=1}^{\infty}\frac{1}{n!}\int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\lambda}...\int_{\lambda_0}^{\lambda}\mathcal{P}[A(\eta_n)A(\eta_{n-1})...A(\eta_1)]d^n\eta$$

where, the *path-ordering symbol*, $\mathcal{P}[A(\eta_n)A(\eta_{n-1}...A(\eta_1)]$, stands for the product of $n$ matrices, $A(\eta_i)$, ordered in such a way that the largest value of $\eta_i$ is on the left and each subsequent value of $\eta_i$ is less that or equal to the previous one. (*3 credits*)
Hint: You may use the fact that there are $n!$ number of $n$-dimensional right-angled triangles (or $n$-simplices) in an $n$-cube.
Remark: This is usually expressed in the following form:

$$P^\mu{}_\rho(\lambda, \lambda_0) = \mathcal{P}\ \mathsf{exp}\left(-\int_{\lambda_0}^{\lambda}\Gamma^\mu{}_{\sigma\rho}\frac{dx^\sigma}{d\eta}d\eta\right)$$

Remarks: Consider a path that starts and ends at the same point ie. a loop. If the connection is metric-compatible, the resulting matrix will just be a Lorentz transformation on the tangent space at that point. This transformation is known as the **holonomy** of the loop. Knowing the holonomy of every possible loop is equivalent to knowing everything about the metric. The corresponding (trace of the) loop parallel propagator in QFT is called a **Wilson loop**. Both Loop Quantum Gravity and String Theory were originally motivated by holonomies in GR and in QCD, respectively.