Physikalisches Institut
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Theoretische Physik

Exercise 07
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Exercises on General Relativity and Cosmology

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–Home Exercises–
Due 1 June 2011

Exercise 7.1: Riemann Tensor ***(14 credits)***

The Christoffel symbols are *not* tensor, and thus are not suitable to describe the physics of curved geometry in a coordinate-invariant way. The only tensor can be constructed from the metric and its first and second derivatives is the Riemann tensor:

$$R^{\lambda}{}_{\mu\nu\kappa} \equiv \partial_\nu \Gamma^{\lambda}{}_{\mu\kappa} - \partial_\kappa \Gamma^{\lambda}{}_{\mu\nu} + \Gamma^{\eta}{}_{\mu\kappa}\Gamma^{\lambda}{}_{\eta\nu} - \Gamma^{\eta}{}_{\mu\nu}\Gamma^{\lambda}{}_{\eta\kappa}$$

Through self-contractions we get the **Ricci tensor** $R_{\mu\kappa} \equiv R^{\lambda}{}_{\mu\lambda\kappa}$ and the **curvature scalar** $R \equiv g^{\mu\kappa} R_{\mu\kappa}$.

(a) Prove the following identity:

$$[D_\nu, D_\kappa] a_\eta = R^{\lambda}{}_{\eta\nu\kappa} a_\lambda$$

(3 credits)

(b) Using the metric, the Riemann tensor can be written in the form:

$$R_{\lambda\mu\nu\kappa} = g_{\lambda\eta} R^{\lambda}{}_{\mu\nu\kappa}$$

Check its symmetry properties:

$$R_{\lambda\mu\nu\kappa} = -R_{\lambda\mu\kappa\nu}$$

$$R_{\lambda\mu\nu\kappa} = +R_{\nu\kappa\lambda\mu}$$

. *(3 credits)*

(c) Calculate the components of $R^{l}{}_{mnk}$, R_{mk} and the curvature scalar R for a space (θ, ϕ) and metric $g_{mn} = \mathrm{diag}(a^2,\ a^2 \sin^2\theta)$ ie. a 2-sphere. *(3 credits)*

(d) Show that for a general two dimensional space, the Riemann tensor takes the form:

$$R_{abcd} = R\ g_{a[c} g_{d]b}$$

. *(3 credits)*

Hint: You may use the fact that the number of independent components of a Riemann tensor in D-dimensions is $D^2(D^2-1)/12$

(e) What is the curvature of a circle with fixed radius R? *(2 credits)*

Exercise 7.2: Bianchi identities ***(6 credits)***

(a) Verify the Bianchi identities:

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0$$

where, the covariant derivative of a generic tensor is given by (extending the results of ex-5.1(b)):

$$X^{\mu\dots}{}_{\nu\dots;\rho} = \partial_\rho X^{\mu\dots}{}_{\nu\dots} + \Gamma^\mu{}_{\rho\sigma} X^{\sigma\dots}{}_{\nu\dots} + \dots - \Gamma^\sigma{}_{\nu\rho} X^{\mu\dots}{}_{\sigma\dots} - \dots$$

(3 credits)

Hint: You may use locally inertial (Riemann normal) coordinates

(b) Contract indices in the above identities to arrive at:

$$\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right)_{;\mu} = 0$$

(3 credits)