Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 08
1 June 2011
SS 2011

Exercises on General Relativity and Cosmology

Priv. Doz. Dr. S. Förste

–Home Exercises–
Due 8 June 2011

Exercise 8.1: Noether's theorem ***(14 credits)***

(a) Consider the action:

$$S = \int d^dx \mathcal{L}(\Phi, \partial_\mu \Phi)$$

with the following large "active" transformation of, both, the position and of the field: $x \to x'$; $\Phi(x) \to \Phi'(x') = \mathcal{F}(\Phi(x))$. Verify the following form of the transformed action: (*2 credits*)

$$S' \equiv \int d^dx\ \mathcal{L}(\Phi'(x), \partial_\mu \Phi'(x)) \overset{\text{verify}}{=} \int d^dx \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left(\mathcal{F}(\Phi(x)),\ \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu \mathcal{F}(\Phi(x))\right)$$

(b) Consider a (active) large Lorentz transformation: $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ and $\Phi'(\Lambda x) = L_\Lambda \Phi(x)$. (where, Λ is the Lorentz transformation matrix and L_Λ is an appropriate representation of the Lorentz group on the field). Show that the transformed action of 8.1(a) takes the following form: (*2 credits*)

$$S' = \int d^dx(L_\Lambda \Phi, (\Lambda^{-1} \cdot \partial(L_\Lambda \Phi))_\mu)$$

(c) Now consider an infinitesimal (small) active transformation:

$$x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a}\ ;\ \Phi'(x') = \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x)$$

where, $\{\omega_a\}$ is a set of infinitesimal parameters. If the **generator** G_a is defined as follows:

$$\delta_\omega \Phi(x) \equiv \Phi'(x) - \Phi(x) \equiv -i\omega_a G_a \Phi(x)$$

Show that it is explicitly given by: (*2 credits*)

$$iG_a \Phi = \frac{\delta x^\mu}{\delta \omega_a}\partial_\mu \Phi - \frac{\delta \mathcal{F}}{\delta \omega_a}$$

(d) Consider an (active) infinitesimal Lorentz transformation:

$$x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu\ ;\ \mathcal{F}(\Phi) \equiv L_\Lambda \Phi \approx \left(1 - \frac{1}{2} i\omega_{\rho\nu} S^{\rho\nu}\right)\Phi$$

Show that the generators of the Lorentz transformations are: (*2 credits*)

$$L^{\rho\nu} = i(x^\rho \partial^\nu - x^\nu \partial^\rho) + S^{\rho\nu}$$

(e) Plug in the infinitesimal transformations of 8.1(c), with varying ω_a, in 8.1(a) and expand Lagrangian to obtain the following *classically* **conserved current**: (*4 credits*)

$$j^\mu_a = \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\partial_\nu \Phi - \delta^\mu_\nu \mathcal{L}\right]\frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\frac{\delta \mathcal{F}}{\delta \omega_a}$$

Hint: Use $\det(1+E) \approx 1 + \mathrm{Tr}(E)$ for small E. Also, assume that the action is invariant under rigid (ie. constant ω_a) transformations

(f) Show that the above current for the Lorentz transformation of 8.1(d) is: (*2 credits*)

$$j^{\mu\nu\rho} = T^{\mu\nu}_c x^\rho - T^{\mu\rho}_c x^\nu + \frac{1}{2} i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} S^{\nu\rho}\Phi$$

where, T_c, is the canonical energy-momentum tensor derived in class.

Exercise 8.2: Various definitions of energy-momentum tensor (***6 credits***)
In general, the **canonical tensor**, $T^{\mu\nu}_c$, is not symmetric. But, since the following modification does not change the classical conservation property:

$$T^{\mu\nu}_B = T^{\mu\nu}_c + \partial_\rho B^{\rho\mu\nu} \quad ; \quad B^{\rho\mu\nu} = -B^{\mu\rho\nu}$$

we can hope to get a *classically* (ie. only for field configurations obeying e.o.m) symmetric $T^{\mu\nu}_B$, called the **Belinfante tensor**. The tensor $B^{\rho\mu\nu}$ is, by no means, unique.

(a) Show that one possible choice of $B^{\rho\mu\nu}$ is: (*2 credits*)

$$B^{\rho\mu\nu} = \frac{1}{4} i \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} S^{\nu\rho}\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \Phi)} S^{\mu\nu}\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} S^{\mu\rho}\Phi\right]$$

Hint: Use the fact that $S^{\mu\nu} = -S^{\nu\mu}$ and use $\partial_\mu j^{\mu\nu\rho} = 0$ of ex-8.1(f) above.

(b) Consider the following Lagrangian for a massive vector field A_μ (in Euclidian spacetime): (*2 credits*)

$$\mathcal{L} = \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} + \frac{1}{2} m^2 A^\alpha A_\alpha$$

where, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial A_\beta A_\alpha$. Write down its Belinfante tensor for $B^{\alpha\mu\nu} = F^{\alpha\mu} A^\nu$

(c) As was said above, the tensor of 8.2(b) will be symmetric only classically. But, there exists another common (*cf.* exercise 0.3(c) for $m = 0$) expression for an identically (ie. classical and otherwise) symmetric energy-momentum tensor of a vector field:

$$\hat{T}^{\mu\nu}_B = F^{\mu\alpha} F^\nu{}_\alpha - \frac{1}{4}\eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} + m^2 \left[A^\mu A^\nu - \frac{1}{2}\eta^{\mu\nu} A^\alpha A_\alpha\right]$$

Show that it coincides with the Belinfante tensor for classical configurations. (*2 credits*)

Remark: The **Hilbert tensor**, derived in class, is an identically symmetric energy-momentum tensor. Note that these are all non-gravitational (ie. matter) energy-momentum tensors. The list of gravitational tensor candidates is longer and contentious.