Physikalisches Institut
Universität Bonn
Theoretische Physik

Mock Exam
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SS 2011

General Relativity and Cosmology
– Mock exam –

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Exercise 1: Overview ***(10 credits)***

Give **short** answers to the following questions:

(a) What is difference in the description of General Relativity as compared to the description of the other fundamental forces? (*1 credit*)

(b) Give 4 experimental evidences for General Relativity. (*2 credits*)

(c) Define a (p,q) tensor. Why are the Christoffel symbols not a tensor? What are they? (*3 credits*)

(d) What are Riemann Normal coordinates. Why does the curvature in general not vanish in Riemann Normal coordinates? (*2 credits*)

(e) Which forms of energy are allowed in a Ricci–flat space? Why? (*2 credits*)

Exercise 2: Electromagnetism in covariant form ***(20 credits)***

In this exercise we consider electromagnetism in its covariant form. In order to do so, we combine the electric potential ϕ and the magnetic potential $\vec{A}$ into the four–potential $A^\mu = (\phi, \vec{A})$. Similarly we define the four–current $j^\mu = (\rho, \vec{j})$ with electric charge density ρ and current density $\vec{j}$. The electro–magnetic field strength derived from A^μ is $F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu$ and in components given by

$$F^{\mu\nu}(t,\mathbf{x}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

(a) By looking at the Lorentz transformation of $F^{\mu\nu}$, argue that neither $\vec{E}$ nor $\vec{B}$ are well–defined vectors. (*1.5 credits*)

(b) Write down a Lorentz-invariant kinetic term, mass term, and source term for A^μ. Argue why the terms are Lorentz invariant. (*2.5 credits*)

(c) Write down a gauge transformation for the vector potential. Which of the above terms are invariant under this transformation, which are not? (*3 credits*)

(d) Define the dual field strength tensor as $\tilde{F}^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. Calculate $\tilde{F}^{\mu\nu}$. *(3 credits)*

(e) Show that Maxwells equations can be written as *(6 credits)*

$$\partial_\mu F^{\mu\nu} = -j^\nu \ , \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0 \ . \tag{1}$$

(f) What is the relation between $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$? Consider the dual version of the above equations:

$$\partial_\mu F^{\mu\nu} = 0 \ , \qquad \partial_\mu \tilde{F}^{\mu\nu} = -j^\nu_{\text{dual}} \ .$$

What do the Maxwell equations for such a configuration look like? What would be the physical consequences if this equation was true? *(2 credits)*

(g) What are the consequences of adding a term $F^{\mu\nu}\tilde{F}_{\mu\nu}$ to the Lagrangian? Explain.
Hint: Use Maxwell's equations (1). *(2 credits)*

Exercise 3: Stereographic Projection ***(25 credits)***

Consider the two–sphere S^2:

$$(x^0, x^1, x^2): \ \ (x^0)^2 + (x^1)^2 + (x^2)^2 = 1 \ .$$

A coordinate chart on $S^2 \backslash \{(1,0,0)\}$ is given by the map $x^a(\xi^i)$, $(a = 0,1,2;\ i = 1,2)$:

$$x^0 = \frac{(\xi^1)^2 + (\xi^2)^2 - 1}{(\xi^1)^2 + (\xi^2)^2 + 1} \ , \qquad x^i = \frac{2\xi^i}{(\xi^1)^2 + (\xi^2)^2 + 1} \ ,$$

which corresponds to a stereographic projection from the north pole onto a plane through an equator.

(a) Take now $\xi^1 = r\cos\phi$ and $\xi^2 = r\sin\phi$ and show that the induces metric satisfies: *(4 credits)*

$$ds^2 = \frac{4}{(1+r^2)^2}(dr^2 + r^2 d\phi^2) \ .$$

(b) Show that the stereographic projection is only invertible on $S^2 \backslash \{(1,0,0)\}$ and determine its inverse. How many coordinate patches are needed to cover S^2? Give these other patches and the transformations between them. *(6 credits)*

(c) Show that the only non vanishing the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ are

$$\Gamma^r_{rr} = -\frac{2r}{1+r^2}, \qquad \Gamma^r_{\phi\phi} = \frac{r(r^2-1)}{1+r^2}, \qquad \Gamma^\phi_{r\phi} = -\frac{r^2-1}{r(1+r^2)}.$$

Hint: you could use the Euler Lagrange formalism. Show that the curve $r(t) = \tan\{\theta/2\}$, $\phi = \phi_0$ (constant) is a geodesic. *(10 credits)*

(d) Compute the Ricci tensor and the curvature scalar. *(5 credits)*

Exercise 4: Parallel transport ***(20 credits)***

In this exercise we will explore geodesics, parallel transport and the Lie derivative.

(a) Give the physical motivation for introducing covariant derivatives in curved spaces? *(2 credits)*

(b) Give the definition of covariant derivatives in terms of locally geodesic coordinates. Show that it ensures the tensor property of $D_\nu V_\mu$, where V^μ is a vector. *(3 credits)*

(c) Why does the metric have to be covariantly costant according to the previous definition? *(2 credits)*

(d) Explain how the covariant derivative helps to define *parallel transport*. Show how the intuitive parallel transport of vectors coincides with the given definition for a one dimensional manifold. *(2 credits)*

(e) Consider the two-sphere with the chart given in Ex.3. Write down the parallel transport equations for a vector T^a, $a = r,\ \phi$ along a curve $(r(t), \phi(t))$. Calculate the transformation of an arbitrary vector which gets parallel transported along the curve $r = R$ (constant), from a point ϕ_0 to $\phi_0 + \alpha$. For which value of R does this curve correspond to a geodesic? Note that in general $T^a(\phi_0) \neq T^a(\phi_0 + 2\pi)$, argue why this result leads to the conclusion that the sphere has non trivial holonomy. Rewrite $T^a(\phi_0 + 2\pi)$ as $T^a(\phi_0 + 2\pi) = M_b^a T^b(\phi_0 + 2\pi)$, compute M_b^a. To which group does this sort of transformations belong? *(6 credits)*

(f) Draw a graph to illustrate the concept of the Lie Derivative. Compute its value along the direction a^μ for a vector V^μ. Give the vector a^μ which ensures that the Lie derivative coincides with the covariant one. *(5 credits)*

Exercise 5: Schwarzschild solution ***(25 credits)***

In this exercise we will examine the Schwarzschild solution for a star. Throughout this exercise you may want to use the formulas provided at the end of the exam.

(a) Given the general static isotropic metric $ds^2 = B(r)dt^2 - A(r)dr^2 + r^2 d\Omega^2$, show that the condition for an asympotically Minkowski space is $A(r) = 1/B(r)$. *(4 credits)*

(b) Consider a slow moving particle in a weak and stationary field of a star. Obtain an expression for g_{00} in terms of the classical gravitational potential $\phi = -GM/r$. Take $|dx/d\tau| \ll |dt/d\tau|$ and a perturbation around flat space $h_{\alpha\beta}$ with $|h_{\alpha\beta}| \ll 1$. *(3 credits)*

(c) Using the result obtained above, derive the Schwarzschild solution by finding the expressions for $A(r)$ and $B(r)$. *(4 credits)*

(d) Employ quasi-Minkowski coordinates

$$x^1 = r \sin\theta \cos\phi\ , \qquad x^2 = r \sin\theta \sin\phi\ , \qquad x^3 = r\cos\theta\ ,$$

to express the metric as

$$ds^2 = B(r)dt^2 - (B(r)^{-1} - 1)r^{-2}(x \cdot dx)^2 - dx^2.$$

Compute the total energy of matter and the gravitational field of the system. Use the expressions given at the end of the exam. Could the result be expected? *(3 credits)*

(e) How many Killing vectors does the obtained metric have? Explain their space–time structure and which conserved quantities they correspond to. (*4 credits*)

(f) Given the Killing vectors $K_1^\mu = (\partial_t)^\mu$ and $K_2^\mu = (\partial_\phi)^\mu$, compute and identify the associated conserved quantities. (*3 credits*)

(g) Use the additional conserved quantity $\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$ in a geodesic to obtain the radial dependence of a particle trajectory in the field of the star in terms of the constants of motion. (*4 credits*)

Exercise 6: FRW Cosmology **(*20 credits*)**

The distribution of matter in the observable universe is homogeneous and isotropic at scales of the order of the Hubble radius. These features are described by the maximally symmetric Robertson-Walker metric:

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right] ,$$

with $a(t)$ the cosmic scale factor. Within a proper rescaling of the coordinates, k can be chosen to be +1, -1 or 0 for spaces with positive, negative and zero spatial curvature, respectively. The non–zero components of the corresponding Ricci tensor are given by:

$$R_{00} = -3\frac{\ddot{a}}{a} , \qquad R_{ij} = -\left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right] g_{ij} .$$

(a) Consider the Einstein equation without cosmological constant

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} .$$

Given the symmetries imposed on the metric, a suitable choice for the energy-momentum tensor is $T_{00} = \rho$, $T_{ij} = -pg_{ij}$. Prove that in such a case, the Einstein equation leads to the Friedmann equations: (*10 credits*)

$$\begin{aligned} \frac{\dot{a}^2}{a^2} + \frac{\dot{a}^2}{a^2} &= \frac{8\pi G}{3}\rho , \\ \frac{\dot{a}}{a} &= -\frac{4\pi G}{3}(\rho + p) . \end{aligned}$$

(b) Use the conservation law of the energy momentum tensor to show that it results in the first law of thermodynamics: (*5 credits*)

$$d\left[a^3(\rho + p)\right] = a^3 dp .$$

(c) For the simple equation of state $p = w\rho$, with w independent of time, what is the value of w for relativistic and non relativistic matter? For which values of w do we have an accelerating universe? Show that the energy density scales as $\rho \propto a^{-3(1+w)}$. (*5 credits*)

Useful formulas

The most general metric tensor that represents an static isotropic gravitational field has $R_{\mu\nu}$ components

$$
\begin{aligned}
R_{rr} &= \frac{B''(r)}{2B(r)} - \frac{1}{4}\left(\frac{B'(r)}{B(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{A'(r)}{A(r)}\right) \\
R_{\theta\theta} &= -1 + \frac{r}{2A(r)}\left(-\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) + \frac{1}{A(r)} \\
R_{\phi\phi} &= \sin^2\theta R_{\theta\theta} \\
R_{\mu\nu} &= 0, \mu \neq \nu \\
R_{tt} &= -\frac{B''(r)}{2B(r)} + \frac{1}{4}\left(\frac{B'(r)}{A(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{B'(r)}{A(r)}\right)
\end{aligned}
$$

The total energy of a gravitational field is given by its perturbation around the flat metric $h_{\mu\nu}$ as

$$
P^0 = -\frac{1}{16\pi G}\int\left(\frac{\partial h_{jj}}{\partial x^i} - \frac{\partial h_{ij}}{\partial^j}\right) n_i r^2 d\Omega \ , \qquad r^2 = x_i x_i \ , \qquad n_i = x_i/r \ .
$$