Physikalisches Institut
Universität Bonn
Theoretische Physik

Mock Exam
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SS 2013

General Relativity and Cosmology

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– Mock Exam –

1.) Overview (15 *Points*)

Give a short answer to the following questions.

(a) Write down Einstein's field equation as well as the Einstein-Hilbert action. What is the definition of the energy-momentum tensor? (**1** *Point*)

(b) What are the Bianchi identities? State the symmetries of $R_{\mu\nu\rho\sigma}$. How many independent components does the Riemann tensor have in four dimensions? (**2** *Points*)

(c) What are Riemann Normal coordinates? Why does the curvature in general not vanish in these coordinates? (**1** *Point*)

(d) State Noether's theorem. What is the relation between the Noether current and the Noether charge? (**1** *Point*)

(e) Use Einstein's equations to show that the Ricci scalar vanishes if the energy-momentum tensor is traceless. What changes if there is in addition a cosmological constant? (**2** *Points*)

(f) What is an affine parameterization of a geodesic? Write down the corresponding geodesic equation. (**1** *Point*)

(g) How are vectors defined on Riemannian manifolds? What are one-forms? (**2** *Points*)

(h) Define the Non-coordinate basis. What are Vielbeins? (**1** *Point*)

(i) What is an affine connection? What does *metric compatibility* mean? (**2** *Points*)

(j) Show that the partial derivative of a tensor does not transform as a tensor under coordinate changes. How is this issue resolved in general relativity? (**2** *Points*)

2.) Equations of motion in curved spacetime (20 *Points*)

The action of a point particle with mass m in a gravitational field is given by

$$S = \frac{1}{2}\int \mathrm{d}\tau \left(\eta^{-1}\dot{x}^2 - \eta m^2\right) = \frac{1}{2}\int \mathrm{d}\tau \left(\eta^{-1}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}g_{\mu\nu} - \eta m^2\right) ,$$

where $\mathrm{d}s^2 = \gamma_{\tau\tau}\mathrm{d}\tau^2$ is an independent metric on the trajectory of the particle and $\eta = \sqrt{-\gamma_{\tau\tau}}$.

(a) Show that the action is invariant under a reparameterization $\tau \mapsto \tau'(\tau)$. (**2** *Points*)

(b) Assume $m \neq 0$. Calculate the equation of motion for η. Insert the solution back into the action and show that one obtains the usual action of a massive point particle. (**3** *Points*)

(c) Now let $m = 0$ and calculate the variation of the action. Which condition do you obtain from the variation with respect to η? Which condition do you obtain from the variation with respect to x^μ?
Hint: To calculate the eom of x^μ, vary first with respect to $\dot{x}^\mu$. Then use partial integration. Do not forget to vary $g_{\mu\nu}$. (**5** *Points*)

Let us now consider the action of the electro-magnetic gauge boson

$$S = \int \mathrm{d}^4x \mathcal{L} = \int \mathrm{d}^4x \sqrt{-g} \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) .$$

(d) Calculate the energy momentum tensor $T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}}$.
Hint: You may use $\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}$. (**4** *Points*)

(e) Consider now flat Minkowski space, i.e. $g_{\mu\nu} = \eta_{\mu\nu}$. Here, one has a symmetry under infinitesimal translations $x^\mu \mapsto x^\mu + \epsilon^\mu$. The conserved Noether-current is the canonical energy momentum tensor $T^{\mathrm{can}}_{\mu\nu}$,

$$T^{\mathrm{can}}_{\mu\nu} = \eta_{\mu\rho} \frac{\delta \mathcal{L}}{\delta(\partial_\rho A_\sigma)} \partial_\nu A_\sigma - \eta_{\mu\nu} \mathcal{L} .$$

Calculate $T^{\mathrm{can}}_{\mu\nu}$. (**4** *Points*)

(f) Show that $T_{\mu\nu}$ from (d) and $T^{\mathrm{can}}_{\mu\nu}$ from (e) differ by a total derivative. (**2** *Points*)

3.) Stereographic Projection (**15** *Points*)

Consider the two-sphere S^2:

$$(x^0, x^1, x^2) : \quad (x^0)^2 + (x^1)^2 + (x^2)^2 = 1 .$$

A coordinate chart on $S^2 \backslash \{(1,0,0)\}$ is given by the map $x^a(\xi^i)$ ($a = 0,1,2$; $i = 1,2$):

$$x^0 = \frac{(\xi^1)^2 + (\xi^2)^2 - 1}{(\xi^1)^2 + (\xi^2)^2 + 1} , \qquad x^i = \frac{2\xi^i}{(\xi^1)^2 + (\xi^2)^2 + 1} ,$$

which corresponds to a stereographic projection from the north pole onto a plane through an equator.

(a) Take now $\xi^1 = r\cos\varphi$ and $\xi^2 = r\sin\varphi$ and show that the induced metric satisfies

$$\mathrm{d}s^2 = \frac{4}{(1+r^2)^2} \left(\mathrm{d}r^2 + r^2 \mathrm{d}\varphi^2 \right) .$$

(**3** *Points*)

(b) Show that the stereographic projection is only invertible on $S^2 \setminus \{(1,0,0)\}$ and determine its inverse. How many coordinate patches are needed to cover S^2? Give these other patches and the transformations between them. (**5** *Points*)

(c) Show that the only non vanishing the Christoffel symbols ${\Gamma^\lambda}_{\mu\nu}$ are

$$
{\Gamma^r}_{rr} = -\frac{2r}{1+r^2}\,, \quad {\Gamma^r}_{\varphi\varphi} = \frac{r(r^2-1)}{1+r^2}\,, \quad {\Gamma^\varphi}_{r\varphi} = -\frac{r^2-1}{r(1+r^2)}\,.
$$

Hint: You could use the Euler Lagrange formalism.
Show that the curve $r(t) = \tan(\theta/2)$, $\varphi = \varphi_0$ (constant) is a geodesic. (**7** *Points*)

4.) Killing vectors and constants of motion (**20** *Points*)

Killing vectors are in one-to-one correspondence to isometries of the metric. Moreover, as we will see here, every Killing vector implies the existence of a conserved quantity.

(a) Show that if the metric is independent of a coordinate x^σ, the corresponding vector ∂_σ is a Killing vector. (**3** *Points*)

(b) Let ξ be a Killing vector, $x^\mu(\lambda)$ a geodesic and $u^\mu(\lambda)$ the four momentum. Show that the quantity $\xi_\mu u^\mu$ is constant along the geodesic.
Hint: Show first that $u^\lambda \nabla_\lambda u^\mu = 0$. (**3** *Points*)

(c) In addition we always have another constant of motion for geodesics. Let λ be an affine parameterization of the geodesic. Show that the geodesic equation implies that the quantity

$$
\epsilon = -g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\,,
$$

is constant along the path. (**2** *Points*)

Consider as an example the Schwarzschild metric

$$
\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)\,.
$$

(d) Write down two Killing vectors of this metric. Show that the corresponding conserved quantities are energy and angular momentum,

$$
E = \left(1-\frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\lambda}\,,
$$
$$
J = r^2\sin^2\theta\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}\,.
$$

(**3** *Points*)

(e) Make use of the spherical symmetry to put $\theta = \frac{\pi}{2}$. Show that any geodesic fulfills

$$
\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + V(r) = \mathcal{E}\,,
$$

where $\mathcal{E} = \frac{1}{2}E^2$ and

$$V(r) = \frac{1}{2}\epsilon - \epsilon\frac{M}{r} + \frac{J^2}{2r^2} - \frac{MJ^2}{r^3}\,.$$

This looks exactly like the equation for a particle of unit mass and 'energy' $\mathcal{E}$ moving in a one-dimensional potential given by $V(r)$. (**5** *Points*)

(f) What is ϵ if the particle is a photon? By analyzing the potential, check whether stable circular orbits exist for photons. (**4** *Points*)

5.) Geodesic deviation equation (**10** *Points*)

Consider a family of geodesics $x^\mu(p,\lambda)$ in some curved spacetime, where λ denotes the affine parameter of the geodesic and p is a label that distinguishes the various geodesics.

(a) Write down the equation that expresses the fact that $x^\mu(p,\lambda)$ is a geodesic for fixed p with affine parameter λ. (**1** *Point*)

(b) Define

$$P^\mu = \frac{\partial x^\mu(p,\lambda)}{\partial p}\,,$$

as well as the relative velocity

$$V^\mu = \frac{\partial P^\mu}{\partial \lambda} + \frac{\partial x^\rho}{\partial \lambda}\Gamma^\mu{}_{\rho\sigma}P^\sigma$$

and the relative acceleration

$$A^\mu = \frac{\partial V^\mu}{\partial \lambda} + \frac{\partial x^\rho}{\partial \lambda}\Gamma^\mu{}_{\rho\sigma}V^\sigma\,.$$

Show that V^μ and A^μ are indeed vectors. What is A^μ in flat space? (**4** *Points*)

(c) Derive the geodesic deviation equation

$$A^\mu = R^\mu{}_{\nu\rho\sigma}\frac{\partial x^\nu}{\partial \lambda}\frac{\partial x^\rho}{\partial \lambda}P^\sigma\,. \tag{1}$$

(**5** *Points*)

6.) Gravitational Waves (**20** *Points*)

Consider gravitational waves propagating in x^3 direction. The metric is given by a perturbation around the Minkowski metric η, i.e. $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$. The solutions to the linearized Einstein equations are linear combinations of solutions of the form

$$h_{\mu\nu} = C_{\mu\nu}\mathrm{e}^{\mathrm{i}k_\alpha x^\alpha}\,, \tag{2}$$

with

$$(C_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

The momenta take the form $k_\mu = (-k, 0, 0, k)$. Consider a family of geodesics $\hat{x}^\mu(p, \lambda)$ in flat space given by

$$\hat{x}^\mu(p, \lambda) = (\lambda, r\cos p, r\sin p, 0) , \tag{3}$$

with p and λ as in problem 5. Now we want to investigate what happens when the gravitational wave passes through this family. In particular, assume that the gravitational wave is very weak. Hence we will assume that the family of geodesics is perturbed by a small amount,

$$x^\mu(p, \lambda) = \hat{x}^\mu(p, \lambda) + \delta x^\mu(p, \lambda) .$$

(a) What does the family (3) describe? (**1** *Point*)

(b) Consider the geodesic deviation equation (1). Take as the metric a gravitational wave and for the family of geodesics the family (3). Expand the geodesic deviation equation to first order in $h_{\mu\nu}$ and $\delta x^\mu(p, \lambda)$. Give the form of the geodesic deviation equation to this order and show that only ${R^\mu}_{00\sigma}$ appears.
Note that products of $h_{\mu\nu}$ and $\delta x^\mu(p, \lambda)$ are considered second order and can therefore be dropped. (**7** *Points*)

(c) Verify that

$${R^\mu}_{00\sigma} = \frac{1}{2}\partial_0\partial_0 {h^\mu}_\sigma$$

to first order in $h_{\mu\nu}$. (**3** *Points*)

(d) Show that the geodesic deviation equation in the form derived in part (b) reduces to

$$\frac{\partial^3 \delta x^\mu}{\partial\lambda^2 \partial p} = \frac{1}{2}\frac{\partial \hat{x}^\nu}{\partial p}\frac{\partial^2 {h^\mu}_\nu}{\partial\lambda^2} . \tag{4}$$

(**2** *Points*)

(e) Show that (4) is solved (among others) by

$$\delta x^\mu = \frac{1}{2}{h^\mu}_\nu \hat{x}^\nu . \tag{5}$$

(**2** *Points*)

(f) By taking appropriate linear combinations of gravitational waves one can make one with

$$h_{11} = -h_{22} = 2C_{11}\cos k\lambda , \qquad h_{12} = h_{21} = 0$$

and another one with

$$h_{12} = h_{21} = 2C_{12}\cos k\lambda , \qquad h_{11} = h_{22} = 0 .$$

For each of these two gravitational waves, use (5) to describe the shape of the family (3) as a function of time as the gravitational wave passes through it. *Hint: To this order of approximation* $t = \lambda$. (**5** *Points*)

Useful relations and definitions:

In the whole exam we use units in which $G_{\mathrm{N}} = c = 1$.

Schwarzschild geometry:
For the general metric

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right) \,,$$

the non-vanishing Christoffel symbols are given by

$$\begin{aligned}
&\Gamma^t{}_{tr} = \frac{A'}{2A}\,, && \Gamma^r{}_{tt} = \frac{A'}{2B}\,, && \Gamma^r{}_{rr} = \frac{B'}{2B}\,,\\
&\Gamma^r{}_{\theta\theta} = -\frac{r}{B}\,, && \Gamma^r{}_{\varphi\varphi} = -\frac{r\sin^2\theta}{B}\,, && \Gamma^\theta{}_{r\theta} = \frac{1}{r}\,,\\
&\Gamma^\theta{}_{\varphi\varphi} = -\sin\theta\cos\theta\,, && \Gamma^\varphi{}_{r\varphi} = \frac{1}{r}\,, && \Gamma^\varphi{}_{\theta\varphi} = \cot\theta\,,
\end{aligned}$$

plus those, one obtains from these by symmetry.