Physikalisches Institut
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Theoretische Physik

Exercise 10
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Exercises on General Relativity and Cosmology

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–Home Exercises–

H 10.1 Physics in curved spacetime and the Einstein-Hilbert action *(10 points)*
In the lecture you have seen that Einstein's equation can be obtained from a variational principle, starting with the action

$$S = S_{\mathrm{EH}} + S_{\mathrm{M}} = \frac{1}{16\pi G_{\mathrm{N}}} \int \mathrm{d}^4 x \sqrt{-g}\, R + S_{\mathrm{M}}\,,$$

where S_{EH} is called the *Einstein-Hilbert action*, S_{M} describes the contribution from matter and G_{N} is Newton's constant.

(a) Derive Einstein's equation of motion by varying the action S with respect to the metric, i.e. show that

$$G_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu}\,.$$

(*2 points*)

(b) In order to derive the result of (a) one has to drop a boundary term coming from the variation with respect to $R_{\mu\nu}$. Explain, why this term should in general be compensated by a boundary term, sometimes referred to as the *Gibbons-Hawking term*. Find the form of such a boundary term. (*2 points*)

(c) How does Einstein's equation change if one adds a cosmological constant term $S_\Lambda = -\frac{1}{8\pi G_{\mathrm{N}}} \int \mathrm{d}^4 x \sqrt{-g}\Lambda$ to the action? Compare your result to splitting the energy momentum tensor into a matter piece and a vacuum piece $T_{\mu\nu} = T^{\mathrm{mat.}}_{\mu\nu} + T^{\mathrm{vac.}}_{\mu\nu}$, where the vacuum energy momentum tensor is that of a perfect fluid with pressure p and density ρ. What is the meaning of the cosmological constant in this picture? (*2 points*)

(d) Assume the matter part to describe a scalar field ϕ in a potential $V(\phi)$, i.e.

$$S_{\mathrm{M}} = \int \mathrm{d}^4 x\, \sqrt{-g} \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right)\,.$$

What is the equation of motion for ϕ? Calculate the energy-momentum tensor. (*2 points*)

(e) Show in general that a theory which is invariant under general coordinate transformations has a covariantly constant energy-momentum tensor, i.e.

$$\nabla_\mu T^{\mu\nu} = 0\,.$$

(*2 points*)

H 10.2 Light deflection (*12 points*)

The motion of a particle around a spherical symmetric and stationary mass distribution of mass M is described by the geodesic equation, where the background metric is chosen to be the Schwarzschild metric

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)\,, \tag{1}$$

where

$$A(r) = \left(1 - \frac{2G_\mathrm{N}M}{r}\right)\,, \qquad B(r) = \left(1 - \frac{2G_\mathrm{N}M}{r}\right)^{-1}\,,$$

r is the distance to the center of mass and the solution is valid for $r > 2M$.

(a) Keeping $A(r)$ and $B(r)$ general for the moment, write down the geodesic equations. (*2 points*)

(b) We can use the spherical symmetry to put $\theta = \frac{\pi}{2}$. Integrate the geodesic equations suitably to get

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{1}{A(r)}\,, \qquad r^2\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} = J = \text{const.}\,, \qquad B(r)\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + \frac{J^2}{r^2} - \frac{1}{A(r)} = -E = \text{const.}\,,$$

where λ is the parameter along the worldline. (*2 points*)

(c) Show that $\mathrm{d}\tau^2 = E\mathrm{d}\lambda^2$. Hence, what does this impose on E if one considers photons or matter? (*1 point*)

(d) Eliminate λ from the integrals of motion obtained in part (b) to obtain a direct relation between r and φ. Show that

$$\varphi = \pm\int \frac{\sqrt{B(r)}\mathrm{d}r}{r^2\sqrt{\frac{1}{A(r)J^2} - \frac{E}{J^2} - \frac{1}{r^2}}}\,. \tag{2}$$

(*2 points*)

Now consider a photon approaching a central mass from infinity with impact parameter b as in figure 1. Denote by r_0 the radius of its closest approach.

(e) Determine E and J in terms of r_0. (*1 point*)

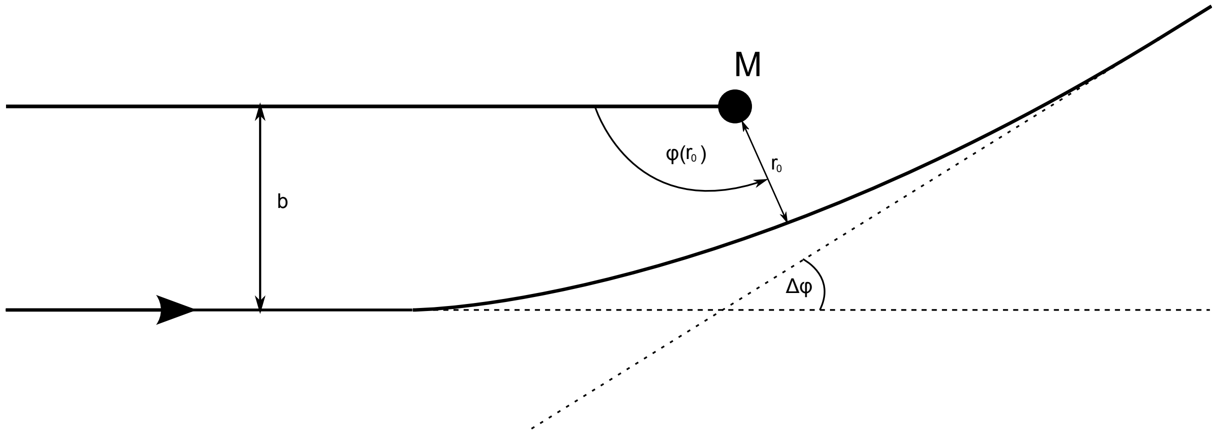

Figure 1: Deflection of a photon approaching a central mass with impact parameter b, $\Delta\varphi = 2\varphi(r_0) - \pi$.

(f) Show that (2) reduces to

$$\varphi(r) = \int_r^\infty \frac{\sqrt{B(r')}}{\sqrt{\frac{r'^2}{r_0^2}\frac{A(r_0)}{A(r')} - 1}} \frac{\mathrm{d}r'}{r'} . \tag{3}$$

(*1 point*)

(g) Use (3) and the approximations for $A(r)$ and $B(r)$ in the Newtonian limit, i.e. $2G_\mathrm{N}M/r \ll 1$, to calculate the deflection angle $\Delta\varphi$. (*3 points*)
Hint: Show, that to lowest order in $2G_\mathrm{N}M/r$*,*

$$\frac{r^2}{r_0^2}\frac{A(r_0)}{A(r)} - 1 = \left[\frac{r^2}{r_0^2} - 1\right]\left[1 - \frac{2G_\mathrm{N}Mr}{r_0(r+r_0)}\right] .$$

The following integrals may be useful

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2-a^2}} = \frac{1}{a}\arccos\frac{a}{x} , \quad \int \frac{\mathrm{d}x}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2 x} , \quad \int \frac{\mathrm{d}x}{(x+a)\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a(x+a)} .$$

H 10.3 Spectral shift (*8 points*)

In the lecture you have already discussed gravitational redshift from the *strong equivalence principle*. Here we want to reconsider this effect in a more formalized way. In order to describe the gravitational effect we consider the spherically symmetric solution to Einstein's equation of a massive object with mass M (Here $G_\mathrm{N} = 1$), which is given by (1), as above. Suppose that a signal is sent from an emitter at a fixed point $(r_\mathrm{E}, \theta_\mathrm{E}, \varphi_\mathrm{E})$, travels along a null geodesic and is received by a receiver at a fixed point $(r_\mathrm{R}, \theta_\mathrm{R}, \varphi_\mathrm{R})$. If t_E is the coordinate time of emission and t_R the coordinate time of reception, then the signal passes from the event with coordinates $(t_\mathrm{E}, r_\mathrm{E}, \theta_\mathrm{E}, \varphi_\mathrm{E})$ to the event with coordiantes $(t_\mathrm{R}, r_\mathrm{R}, \theta_\mathrm{R}, \varphi_\mathrm{R})$.

(a) Draw a spacetime diagram illustrating these events. (*1 point*)

(b) Let λ denote an affine parameterization of the null geodesic, with $\lambda_{\mathrm{E/R}}$ being the point of emission/reception, respectively. Show that

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \left[\left(1 - \frac{2M}{r}\right)^{-1} g_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \right]^{\frac{1}{2}} .$$

(2 points)

(c) Use the above results to argue that

$$\Delta t_{\mathrm{E}} = \Delta t_{\mathrm{R}} ,$$

where $\Delta t = t^{(1)} - t^{(2)}$ denotes the coordiante time difference between two signals 1 and 2. *(1 point)*

(d) A clock situated at the position of an observer records proper time τ instead of coordinate time t. Find a relation between those two notions of time and conclude that

$$\frac{\Delta \tau_{\mathrm{R}}}{\Delta \tau_{\mathrm{E}}} = \left[\frac{1 - 2M/r_{\mathrm{R}}}{1 - 2M/r_{\mathrm{E}}} \right]^{\frac{1}{2}} .$$

(1 point)

(e) Suppose the emmiter is pulsating at a frequency $\nu_{\mathrm{E}} = \frac{n}{\Delta \tau_{\mathrm{E}}}$, i.e. there are n pulses per proper time interval $\Delta \tau_{\mathrm{E}}$. Similar expressions hold for the receiver. Find the relation between the two frequencies $\nu_{\mathrm{E}}/\nu_{\mathrm{R}}$. Expand this relation for $r_{\mathrm{E}}, r_{\mathrm{R}} \gg 2M$ and discuss what happens if the emitter (receiver) is nearer to the massive object than the receiver (emitter). Compare your results to the one found in the lecture,

$$\frac{\Delta \nu}{\nu} = gz ,$$

where z is the distance between emitter and receiver (note that here we are working in units where $c = 1$). *(3 points)*

H 10.4 Inner Schwarzschild solution *(10 points)*

Since the Schwarzschild solution as given above is only valid outside of the spherically symmetric mass distribution, let us here try to find a continuation which holds inside of the massive object (e.g. a star). We will do so by modelling the object as made of an ideal fluid[1], i.e. its engery-momentum tensor is given by

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu} .$$

For the metric we take the spherically symmetric, static[2] Ansatz ($c = 1$)

$$\mathrm{d}s^2 = -\mathrm{e}^{\nu(r)}\mathrm{d}t^2 + \mathrm{e}^{\lambda(r)}\mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right) .$$

[1]That is we ignore thermodynamic effects, such as heat conduction and viscosity.

[2]Hence ignoring radial matter currents.

(a) Show that for matter to be at rest in these coordinates, u^μ fulfills

$$(u^\mu) = \begin{pmatrix} e^{-\nu/2} & 0 & 0 & 0 \end{pmatrix} .$$

(1 point)

Plugging in the well-known components of the Ricci tensor into the Einstein equation, we arrive at the set of differential equations

$$-\kappa\rho = -e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] - \frac{1}{r^2}\,, \tag{4a}$$

$$\kappa p = e^{-\lambda}\left[\frac{\nu'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2}\,, \tag{4b}$$

$$\kappa p = e^{-\lambda}\left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{\nu' - \lambda'}{2r}\right]\,, \tag{4c}$$

where $\kappa = 8\pi G_{\mathrm{N}}$.

(b) Show, that the conservation of the energy-momentum tensor, $(\nabla_\mu T)^{\mu\nu} = 0$, implies

$$p' = -\frac{\nu'}{2}(p+\rho)\,. \tag{5}$$

Since this equation is a consequence of the field equations (4), it can be used in place of one of the three equations. *(2 points)*

(c) Show that the solution of (4a) is given by

$$e^{-\lambda(r)} = 1 - \frac{2m(r)}{r} + \frac{C}{r}\,, \qquad m(r) = \frac{\kappa}{2}\int_0^r \rho(r')r'^2\mathrm{d}r'\,,$$

where C is some integration constant. The requirement of g^{rr} to be finite at $r=0$ implies $C=0$. *(3 points)*

We still have the freedom to choose an equation of state $f(\rho,p)=0$ for the fluid. For simplicity we will here assume a constant rest-mass density

$$\rho = \text{const.}\,. \tag{6}$$

Note that this equation of state certainly does not give a good stellar model. A constant mass density is a first approximation only for small stars in which the pressure is not too large. The spherically symmetric, static solution with the equation of state (6) is called the **interior Schwarzschild solution**.
Using this equation of state, the solution of (4a) simplifies to

$$e^{-\lambda(r)} = 1 - \frac{2m(r)}{r}\,, \qquad m(r) = \frac{\kappa\rho r^3}{6}\,, \tag{7}$$

and (5) can be integrated to give

$$p + \rho = Be^{-\nu/2}\,, \tag{8}$$

with an integration constant B. Now we can choose a linear combination of the equations (4) as the third independent differential equation.

(d) Show that the linear combination $-$(4a)+(4b) implies

$$\left[\mathrm{e}^{\nu/2}\left(1-\frac{2m}{r}\right)^{-\frac{1}{2}}\right]' = \frac{\kappa B r}{2\left(1-\frac{2m}{r}\right)^{\frac{3}{2}}}\,.$$

Use this to find the solution

$$\mathrm{e}^{\nu/2} = \frac{r^3}{4m}\kappa B - D\sqrt{1-\frac{2m}{r}}\,, \tag{9}$$

where D is another constant of integration. *(4 points)*

In total equations (7), (8) and (9) provide us with the general solution for a constant mass density. They contain two constant of integration, B and D, which can be determined by matching the interior Schwarzschild solution to the outer Schwarzschild solution. This is done by simply demanding continuity of the metric $g_{\mu\nu}$ and its derivatives $\partial_\sigma g_{\mu\nu}$. In the present case this means continuity of e^ν, e^λ and of p ($p=0$ at $r=r_0$).
Using the outer Schwarzschild solution as given in (1), this explicitly means

$$M = \frac{1}{6}\kappa\rho r_0^3\,, \qquad D = \frac{1}{2} \qquad \text{and} \qquad B = \rho\sqrt{1-\frac{2M}{r_0}}\,.$$

In summary we get the following result for the spherically symmetric gravitational field of a star with mass density $\rho=$ const. and radius r_0:

$$\mathrm{d}s^2 = \begin{cases} -\left[\frac{3}{2}\sqrt{1-\frac{2M}{r_0}}-\frac{1}{2}\sqrt{1-\frac{2m}{r}}\right]^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{2m}{r}} + r^2\mathrm{d}\Omega^2 & \text{for} \quad r\le r_0\,,\\ -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{2M}{r}} + r^2\mathrm{d}\Omega^2 & \text{for} \quad r> r_0\,,\end{cases}$$

$$\rho = \begin{cases} \frac{6M}{\kappa r_0^3} & \text{for} \quad r\le r_0\,,\\ 0 & \text{for} \quad r> r_0\,,\end{cases}$$

$$p = \begin{cases} \frac{6M}{\kappa r_0^3}\left(\frac{\sqrt{1-\frac{2m}{r}}-\sqrt{1-\frac{2M}{r_0}}}{3\sqrt{1-\frac{2M}{r_0}}-\sqrt{1-\frac{2m}{r}}}\right) & \text{for} \quad r\le r_0\,,\\ 0 & \text{for} \quad r> r_0\,,\end{cases}$$

where $m = M\left(\frac{r}{r_0}\right)^3$, $M=\frac{4\pi}{3}G_{\mathrm{N}}\rho r_0^3$ and $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2$.