Physikalisches Institut
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Theoretische Physik

Exercise 11
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SS 2013

Exercises on General Relativity and Cosmology

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http://www.th.physik.uni-bonn.de/people/forste/exercises/ss2013/gr

Your solutions to the exercises on this sheet do not affect your admittance to the final exam. You may hand in solutions to your tutors until Wednesday, 10th of july. The exercises will be discussed in the tutorials in week 29.

–Home Exercises–

H 11.1 Homogeneous and isotropic universe *(0 points)*

A four-dimensional homogeneous, isotropic universe is described by the *Robertson-Walker metric*,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\right], \tag{1}$$

where $k \in \{-1, 0, +1\}$ and $a(t)$ is called the *scale factor*. A long but straightforward calculation yields the following non-vanishing components of the Ricci tensor

$$R_{tt} = -3\frac{\ddot{a}}{a}, \qquad R_{ij} = \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2}\right) g_{ij}.$$

The energy momentum tensor of the fields in the universe respecting the symmetries is that of a perfect fluid: $(T^\mu{}_\nu) = \mathrm{diag}(-\rho, p, p, p)$.

(a) Derive the *Friedmann equations*

$$\begin{aligned}\frac{\ddot{a}}{a} &= -\frac{4\pi G_\mathrm{N}}{3}(\rho + 3p), \\ \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G_\mathrm{N}}{3}\rho - \frac{k}{a^2}.\end{aligned} \tag{2}$$

(b) Show that (2) implies the first law of thermodynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho a^3\right) = -p\frac{\mathrm{d}}{\mathrm{d}t}a^3.$$

(c) In order to solve (2) one further needs an equation of state $p = p(\rho)$. Assume the simple relation

$$p = \omega\rho$$

and show that in the flat case ($k = 0$) the scale factor fulfills

$$a \propto \begin{cases} t^{\frac{2}{3(1+\omega)}} & \omega \neq -1\,, \\ \mathrm{e}^t & \omega = -1 \end{cases} .$$

Under which circumstances do the Friedmann equations describe an accelerated expansion of the universe?

(d) Denote by a subscript 0 today's quantities. Define the *Hubble parameter* $H(t)$, today's *critical density* ρ_c and the *density parameter* Ω as

$$H(t) = \frac{\dot{a}}{a}\,, \qquad \rho_\mathrm{c} = \frac{3H_0^2}{8\pi G_\mathrm{N}}\,, \qquad \Omega = \frac{\rho}{\rho_\mathrm{c}}\,.$$

Rewrite the second equation of (2) as

$$\Omega_0 - 1 = \frac{k}{a_0^2 H_0^2}\,.$$

Interprete this equation.

(e) Define time t such that $a(t = 0) = 0$, normalize $a(t_0) = a_0 = 1$ and show that the age of the universe t_0 is given by

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{\mathrm{d}a}{\sqrt{1 - \Omega_0 + \Omega_0 a^{-1-3\omega}}}\,.$$

Calculate t_0 for a matter-dominated and radiation-dominated universe for the open ($k = -1$), flat ($k = 0$) and closed ($k = +1$) case. The following integrals might be useful

$$\int_0^1 \sqrt{\frac{x}{1+x}}\mathrm{d}x = \sqrt{2} - \ln(1+\sqrt{2})\,, \qquad \int_0^1 \sqrt{\frac{x}{1-x}}\mathrm{d}x = \frac{\pi}{2}\,,$$

$$\int_0^1 \frac{x\mathrm{d}x}{\sqrt{1+x^2}} = 1 \qquad \int_0^1 \frac{x\mathrm{d}x}{\sqrt{1-x^2}} = \sqrt{2} - 1\,.$$

H 11.2 Hubble's law *(0 points)*

(a) Show that the redshift z can be expressed by the scale factor as

$$z = \frac{a(t_0)}{a(t_1)} - 1\,.$$

(b) Consider a Taylor expansion of the scale factor $a(t)$ around $a(t_0)$ and use this to show that

$$z = H_0(t_0 - t) + \frac{1}{2}(2 + q_0)H_0^2(t_0 - t)^2 + \ldots\,,$$

where $q = -a\ddot{a}/\dot{a}^2$ denotes the so-called *deceleration parameter*.

(c) Obtain an expression for $t_0 - t$ by inverting the result of (b).

(d) Consider radial photon propagation in order to find the following expression for r,

$$r = a^{-1}(t_0)\left((t_0 - t) + \frac{1}{2}H_0(t_0 - t)^2 + \dots\right).$$

(e) Use the luminosity distance $d_{\mathrm{L}} = a_0 r(1+z)$ to find Hubble's law in terms of measurable quantities

$$H_0 d_{\mathrm{L}} = z + \frac{1}{2}(1 - q_0)z^2 .$$