Physikalisches Institut
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Theoretische Physik

Exercise 2
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Exercises on General Relativity and Cosmology

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H 2.1 Contra- and covariant tensors *(11 points)*

The components x^μ which we used in the previous exercise sheet are called *contravariant* coordinates of the four-vector x, which itself is an element of the tangent space $T_p(M)$ at a point p in a manifold[1] M. If M is an n-dimensional manifold, then $T_p(M)$ is an n-dimensional vector space for which we can write down a basis $\{\hat{e}_{(0)}, \dots, \hat{e}_{(n-1)}\}$. Consequently each vector v in $T_p(M)$ can be written as $v = v^\mu \hat{e}_{(\mu)}$ where (here and always) we use *Einstein's sum convention*.

The *dual vector space* $T_p^*(M)$ is the space of linear maps $T_p(M) \to \mathbb{R}$ and has a basis $\{\hat{e}^{(0)}, \dots, \hat{e}^{(n-1)}\}$, defined by

$$\hat{e}^{(\mu)}(\hat{e}_{(\nu)}) = \delta^\mu_\nu \,.$$

Here we want to clarify the relation of upper and lower indices that appear in special relativity from a more general point of view. Since the following will be true in general, let us consider a general n-dim. vector space V with basis $\{\hat{e}_{(0)}, \dots, \hat{e}_{(n-1)}\}$ and its dual vector space V^* with basis $\{\hat{e}^{(0)}, \dots, \hat{e}^{(n-1)}\}$, while keeping in mind that we will encounter the case $V = T_p(M)$, $V^* = T_p^*(M)$ in the context of general relativity. Given a symmetric *bilinear form* $\beta : V \times V \to \mathbb{R}$ (i.e. β is a function which is linear in both its arguments), we can define an *isomorphism* $\phi : V \to V^*$ by setting

$$\phi(v) = \beta(v, \cdot) \,,$$

as well as a symmetric *bilinear form* $\beta^* : V^* \times V^* \to \mathbb{R}$ given by

$$\beta^*(\phi(v), \phi(w)) = \beta(v, w) \,.$$

We introduce the notation

$$\beta_{\mu\nu} = \beta(\hat{e}_{(\mu)}, \hat{e}_{(\nu)}) \,, \qquad \beta^*(\hat{e}^{(\mu)}, \hat{e}^{(\nu)}) = \beta^{*\mu\nu} \,.$$

(a) Given a vector $v = v^\mu \hat{e}_{(\mu)} \in V$, show that its dual $\tilde{v}$ is given by

$$\tilde{v} = \tilde{v}_\mu \hat{e}^{(\mu)} \,, \qquad \tilde{v}_\mu = \beta_{\mu\nu} v^\nu \,.$$

Given the existence of the isomorphism ϕ, the coordinates $\tilde{v}_\mu$ are then called *covariant* coordinates of the vector v (and one often supresses the tildes). (*1 point*)

[1] Manifolds will become important in the framework of general relativity later. For now we just regard them as some kind of spaces.

(b) Show that

$$(\beta_{\mu\nu}) = (\beta^{*\mu\nu})^{-1} \, .$$

(2 points)

If V is the tangent space of a *Riemannian manifold* M, one is equipped with a canonical choice for the bilinear form β, namely the metric. In the case of special relativity (i.e. $M = \mathbb{R}^{3,1}$), which we want to focus on from now on, the metric is

$$\eta_{\mu\nu} = \begin{cases} -1 & \text{for} \quad \mu = \nu = 0 \\ 1 & \text{for} \quad \mu = \nu = 1,2,3 \\ 0 & \text{for} \quad \mu \neq \nu \end{cases} \, .$$

Hence, the *covariant components* of a vector v are given by

$$v_\mu = \eta_{\mu\nu} v^\nu \, .$$

In general the metric is used to lower and raise indices, where raising indices is defined analogously through the help of $\eta^{\mu\nu}$ (from now on we suppress the *), which are, as you have shown in (b), defined to be the components of the (matrix) inverse of η, i.e.

$$(\eta^{\mu\nu}) = (\eta_{\mu\nu})^{-1} \, .$$

(c) Show that the components of the inverse matrix of a Lorentz transformation Λ fulfill ${(\Lambda^{-1})^\mu}_\nu = {\Lambda_\nu}^\mu$. *(2 points)*

(d) How do the covariant coordinates x_μ of a vector x transform under a Lorentz transformation? *(2 points)*

(e) Show that $\partial_\mu = \frac{\partial}{\partial x^\mu}$ transforms as a covariant vector component and $\partial^\mu = \frac{\partial}{\partial x_\mu}$ as a contravariant vector component under Lorentz transformations. *(2 points)*

Generalizing the above definition of covariant vector components, let us define a (k,l)-tensor T as a multilinear map

$$T : \underbrace{V^* \times \cdots \times V^*}_{k \text{ times}} \times \underbrace{V \times \cdots \times V}_{l \text{ times}} \to \mathbb{R} \, ,$$

with components

$$T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} = T\left(\hat{e}^{(\mu_1)}, \ldots, \hat{e}^{(\mu_k)}, \hat{e}_{(\nu_1)}, \ldots, \hat{e}_{(\nu_l)}\right) \, .$$

Note that the space of (k,l) tensors forms a vector space, namely

$$\underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l \text{ times}} \, ,$$

where $\otimes$ denotes the tensor product (of vector spaces).

(f) Show that the components of a (k,l) tensor T transform under a Lorentz transformation Λ as

$$T^{\mu'_1\dots\mu'_k}{}_{\nu'_1\dots\nu'_l} = \left(\Lambda^{-1}\right)^{\nu_1}{}_{\nu'_1}\dots\left(\Lambda^{-1}\right)^{\nu_l}{}_{\nu'_l}\Lambda^{\mu'_1}{}_{\mu_1}\dots\Lambda^{\mu'_k}{}_{\mu_k}T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_l}\,.$$

(*1 point*)

(g) Show that the components of the metric $\eta_{\mu\nu}$ transform as a $(0,2)$-tensor and that the d'Alembert operator $\Box = \partial^\mu\partial_\mu$ is a scalar. (*1 point*)

H 2.2 Electromagnetism (*9 points*)

Maxwell's equations can be written by using Lorentz-Heaviside units and $c=1$ as

$$\vec{\nabla}\cdot\vec{E} = \rho\,,\qquad \vec{\nabla}\times\vec{B} = \frac{\partial\vec{E}}{\partial t} + \vec{j}\,,\qquad \vec{\nabla}\cdot\vec{B} = 0\,,\qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\,.$$

We can make the Lorentz covariance explicit by introducing an antisymmetric tensor $F^{\mu\nu} = -F^{\nu\mu}$ defined by

$$F^{0i} = E^i \qquad F^{ij} = \sum_{k=1}^{3}\epsilon^{ijk}B^k\,.$$

(a) Show that

$$\partial_\mu F^{\mu\nu} = -J^\nu \quad\text{and}\quad \partial_{[\mu}F_{\nu\lambda]} = 0$$

reproduces Maxwell's equations, where $(J^\mu) = (\rho,\vec{j})$ and $_{[\dots]}$ denotes total antisymmetrization of the indices. (*3 points*)

(b) Use the transformation properties of the tensor $F_{\mu\nu}$ to deduce the transformation behaviour of $\vec{E}$ under a boost along the x^1-direction. (*3 points*)

(c) Verify that

$$f^\mu \equiv \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = eF^\mu{}_\nu\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$

is the correct equation of the electromagnetic four-force f^μ acting on a particle of charge e and mass m. Do this by evaluating in the rest frame of the charged particle. Moreover, show that it reproduces the Lorentz force, i.e. show

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = e\left(\vec{E} + \vec{v}\times\vec{B}\right)\,.$$

(*3 points*)