Physikalisches Institut
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Theoretische Physik

Exercise 3
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Exercises on General Relativity and Cosmology

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H 3.1 Some concrete tensor algebra *(4 points)*

Given the components of a $(2,0)$-tensor X as well as the components of a vector V,

$$(X^{\mu\nu}) = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}, \qquad (V^\mu) = \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2 \end{pmatrix},$$

compute

(a) $X^\mu{}_\nu$,

(b) $X_\mu{}^\nu$,

(c) $X^{(\mu\nu)}$,

(d) $X^{[\mu\nu]}$,

(e) $X^\mu{}_\mu$,

(f) $V^\mu V_\mu$,

(g) $V_\mu X^{\mu\nu}$.

H 3.2 Energy-momentum tensor *(16 points)*

Given a collection of charged particles with positions $\vec{x}_n(t)$ and charges e_n we define electric charge density ρ and current density $\vec{j}$ as

$$\rho(\vec{x},t) = \sum_n e_n \delta^3(\vec{x}-\vec{x}_n(t)), \qquad \vec{j}(\vec{x},t) = \sum_n e_n \dot{\vec{x}}_n(t)^\nu \delta^3(\vec{x}-\vec{x}_n(t)).$$

Similarly, we define the charge density for the four-momentum p^μ, the *energy-momentum tensor*, as

$$T^{\mu\nu} = \sum_n p_n^\mu(t) \dot{x}_n^\nu(t) \delta^3(\vec{x}-\vec{x}_n(t)).$$

(a) Check that the $T^{\mu\nu}$ transform as components of a $(2,0)$-tensor. *(1 point)*

(b) Show that the energy-momentum tensor is only conserved up to a force density G^μ which vanishes for free particles

$$\partial_\nu T^{\mu\nu} = G^\nu \, .$$

(*2 points*)

(c) Check that for the electromagnetic force from exercise H2.2(c),

$$f^\mu \equiv \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = eF^\mu{}_\nu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \, ,$$

G is given by

$$G^\mu = F^\mu{}_\nu J^\nu \, .$$

(*1 point*)

(d) To obtain a conserved energy-momentum tensor, we have to include the contribution of the electromagnetic field itself

$$T^{\mu\nu}_{\text{em}} = F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \, .$$

Show that $\partial_\nu T^{\mu\nu}_{\text{em}}$ cancels G^μ introduced in point (b). Thus, $T^{\mu\nu}_{\text{tot.}} = T^{\mu\nu} + T^{\mu\nu}_{\text{em}}$ is conserved. (*2 points*)

(e) Show that the total momentum

$$P^\mu = \int \mathrm{d}^3x \; T^{\mu 0}_{\text{tot.}}(\vec{x}, t)$$

is a conserved quantity. (*2 points*)

Now we want to consider the energy-momentum tensor of a *perfect fluid*. A comoving observer will, by definition, see his surroundings as isotropic. In this frame the energy-momentum tensor is given by

$$\left(\tilde{T}^{\mu\nu}\right) = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \, ,$$

where ρ is the density and p the pressure of the fluid.

(f) Calculate the components of energy-momentum tensor $T^{\mu\nu}$ for an observer at rest. Assume the comoving observer's velocity to be $\vec{v}$. (*3 points*)

(g) Show that $T^{\mu\nu}$ can also be written as

$$T^{\mu\nu} = (p + \rho) U^\mu U^\nu + p\eta^{\mu\nu} \, ,$$

where U^μ are the components of the four-velocity of the fluid. (*2 points*)

(h) From the nonrelativistic limit of the conservation of the energy momentum tensor, $\partial_\mu T^{\mu\nu}$, deduce *Euler's equations*

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0\,,$$
$$\rho \left[\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right] = -\vec{\nabla} p\,.$$

Hint: The nonrelativistic limit is given by $(U^\mu) = (1, v^i)$, $|v^i| \ll 1$, $p \ll \rho$. *Project the equation into pieces along and orthogonal to the four-velocity by contraction with* U_ν *and* $P^\sigma{}_\nu = \delta^\sigma_\nu + U^\sigma U_\nu$ *respectively.* (*3 points*)