Exercises on General Relativity and Cosmology

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H 4.1 Differential Forms (I) *(12 points)*

In this exercise we want to discuss a special class of tensors called *differential forms*. They are of special importance in theoretical physics since they are useful tools in the formulation of both general relativity and gauge theories. Let us first recapitulate the vector space spanned by (p,q) tensors, which is the vector space of multilinear maps

$$\underbrace{V^* \times \cdots \times V^*}_{p \text{ times}} \times \underbrace{V \times \cdots \times V}_{q \text{ times}} \to \mathbb{R}\,,$$

denoted by $L^{p+q}(\underbrace{V^*,\ldots,V^*}_{p \text{ times}},\underbrace{V,\ldots,V}_{q \text{ times}};\mathbb{R})$. One can show, that there is a natural isomorphism from this vector space to the tensor product space

$$\underbrace{V \otimes \cdots \otimes V}_{p \text{ times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{q \text{ times}},$$

where the **tensor product** of vector spaces is defined as follows:
Let V,W be vector spaces of dimension d_V and d_W and with bases $\{v_{(\alpha)}\}$ and $\{w_{(i)}\}$ respectively. Then a basis of the $d_{V\otimes W} = d_V \cdot d_W$ dimensional vector space $V \otimes W$ is given by the set[1] $\{v_{(\alpha)} \otimes w_{(i)}\}$, i.e.

$$\boxed{V \otimes W = \{a^{\alpha i} v_{(\alpha)} \otimes w_{(i)} \mid a^{\alpha i} \in \mathbb{R}\}\,.}$$

An element of $V \otimes W$ is thus specified by its components $a^{\alpha i}$, which can be pictured as a matrix

$$\left(a^{\alpha i}\right) = \begin{pmatrix} a^{11} & \ldots & a^{1 d_W} \\ \vdots & \ddots & \vdots \\ a^{d_V 1} & \ldots & a^{d_V d_W} \end{pmatrix}.$$

Note that altough the basis elements of the tensor product space are products of the basis elements of the factors, a general element in $V \otimes W$ cannot be written as a single tensor product of an element of V and an element of W. Those element for which this is possible are called *pure tensors*. Hence every tensor is a linear combination of pure tensors. The canonical example for the appearance of tensor product spaces is entanglement in quantum mechanics: Consider two spins which can each be up or down, the states being labeled by $|\uparrow\rangle_{1,2}$ and $|\downarrow\rangle_{1,2}$. Then states of the 2-spin system are tensor products of the individual spin states.

[1] Here $v_{(\alpha)} \otimes w_{(i)}$ is just a formal way of writing an ordered pair.

(a) In this example what are the pure tensors? Write down a state which is not pure. (*1 point*)

Let now $V = T_p(M)$ be the tangent space at a point p, and $V^* = T_p^*(M)$ the corresponding dual vector space, namely the cotangent space at that point. Let V and V^* have bases $\{\hat{e}_{(\mu)}\}$ and $\{\hat{\theta}^{(\mu)}\}$ with $\mu = 0, \ldots, d-1$. A **differential form** A_r of order r (or r**-form**) is a totally antisymmetric $(0, r)$-tensor

$$\boxed{A_r = \frac{1}{r!} A_{\mu_1 \ldots \mu_r} \hat{\theta}^{(\mu_1)} \wedge \cdots \wedge \hat{\theta}^{(\mu_r)}\,,}$$

where we have used the **wedge product**, given by

$$\hat{\theta}^{(\mu_1)} \wedge \cdots \wedge \hat{\theta}^{(\mu_r)} = \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma)\, \hat{\theta}^{(\mu_{\sigma(1)})} \otimes \cdots \otimes \hat{\theta}^{(\mu_{\sigma(r)})}\,.$$

Hence the wedge product is antisymmetric, $\hat{\theta}^{\mu_1} \wedge \hat{\theta}^{\mu_2} = -\hat{\theta}^{\mu_2} \wedge \hat{\theta}^{\mu_1}$ and it extends to arbitrary forms,

$$\begin{aligned} A_p \wedge B_q &= \frac{1}{p!}\frac{1}{q!} A_{\mu_1 \ldots \mu_p} B_{\nu_1 \ldots \nu_q} \hat{\theta}^{(\mu_1)} \wedge \cdots \wedge \hat{\theta}^{(\mu_p)} \wedge \hat{\theta}^{(\nu_1)} \wedge \cdots \wedge \hat{\theta}^{(\nu_q)} \\ &= \frac{1}{(p+q)!} \left(A_p \wedge B_q\right)_{\mu_1 \ldots \mu_{p+q}} \hat{\theta}^{(\mu_1)} \wedge \cdots \wedge \hat{\theta}^{(\mu_{p+q})}\,. \end{aligned}$$

So the components of the product form are given by

$$\boxed{\left(A_p \wedge B_q\right)_{\mu_1 \ldots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \ldots \mu_p} B_{\mu_{p+1} \ldots \mu_{p+q}]}\,.}$$

The set of p-forms on the vector space V forms a vector space denoted by $A^k(V)$.

(b) How does a $(d+1)$-form look like if the vector space V is d-dimensional? (*1 point*)

(c) How many independent components does a p-form have if the vector space V is d-dimensional? Compare this to the number of independent components of a *symmetric* (k, l)-tensor with $k + l = p$. (*2 points*)

A very important concept when dealing with forms is the **exterior derivative**, $\mathrm{d} : A^p(V) \to A^{p+1}(V)$ and its action on a p-form A_p is given by

$$\begin{aligned} \mathrm{d}A_p &= \mathrm{d}\left(\frac{1}{p!} A_{\mu_1 \ldots \mu_p} \hat{\theta}^{\mu_1} \ldots \hat{\theta}^{\mu_p}\right) \\ &= \frac{1}{p!} \partial_\rho A_{\mu_1 \ldots \mu_p} \hat{\theta}^{\rho} \wedge \hat{\theta}^{\mu_1} \wedge \cdots \wedge \hat{\theta}^{\mu_p}\,, \end{aligned}$$

so the components of the resulting $(p+1)$-form are

$$\boxed{\left(\mathrm{d}A_p\right)_{\mu_1 \ldots \mu_{p+1}} = (p+1) \partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]}\,.}$$

The easiest and most prominent example of exterior derivatives are *total derivatives* $\mathrm{d}f$ of **0-forms** (functions) $f : V \to \mathbb{R}$.

(d) Verify that the result of the exterior derivative indeed transforms as a tensor under Lorentz transformations. Furthermore, show that $\mathrm{d}^2 = 0$ and that the exterior derivative satisfies a Leibniz rule,

$$\mathrm{d}\left(A_p \wedge B_q\right) = \mathrm{d}A_p \wedge B_q + (-1)^p A_p \wedge \mathrm{d}B_q \,.$$

(*3 points*)

Now given a metric g on the d-dimensional manifold M[2], we can assign a $(d-p)$-form $(*A)_{d-p}$ to a p-form A_p, which has components

$$\boxed{(*A)_{\mu_1\ldots\mu_{d-p}} = \frac{1}{p!}\sqrt{|\det g|}\;\epsilon_{\mu_1\ldots\mu_d} g^{\mu_{d-p+1}\nu_1}\ldots g^{\mu_d\nu_p} A_{\nu_1\ldots\nu_p}\,.}$$

This operation is called **Hodge-star**. You have seen in the lecture that for the case $g = \eta$ this indeed transforms as a tensor. Later we will show that $\sqrt{|\det g|}\;\epsilon_{\mu_1\ldots\mu_d}$ also transforms as a tensor for general manifolds.

(e) Compute the action of $**$. (*1 point*)

(f) Specialise to three-dimensional Euclidean space $\mathbb{R}^3$. Consider a scalar function $\phi(x)$ and a vector field $\vec{u}(x)$ and express the usual operations grad, curl and div in form language. Derive the well-known identities

  (i) $\mathrm{curl}\,\mathrm{grad}\,\phi = 0$,
  (ii) $\mathrm{div}\;\mathrm{curl}\,\vec{u} = 0$.
  (iii) Let $\vec{v}$ be another vector field. Express the cross product $\vec{u}\times\vec{v}$ by forms.

(*3 points*)

(g) Rewrite Maxwell's equations in form language. (*1 point*)

H 4.2 Euclidean Metrics (*4 points*)

We consider Euclidean spacetime in different coordinate systems. Calculate the metric (i.e. the line element) in

(a) 3d spherical coordinates, (*2 points*)

(b) 2d polar coordinates. (*2 points*)

[2] As on sheet 2 we regard manifolds as some kind of spaces which have a tangent space at each point $p \in M$. As an easy example you can think of $M = \mathbb{R}^{3,1}$, for which the metric is η.