Physikalisches Institut
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Theoretische Physik

Exercise 5
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Exercises on General Relativity and Cosmology

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http://www.th.physik.uni-bonn.de/people/forste/exercises/ss2013/gr

–Home Exercises–

H 5.1 Matrix Identity *(6 points)*

Show the identity

$$\det A = \mathrm{e}^{\mathrm{tr}\log A}\,,$$

for general real valued $d \times d$ matrix A (for which log is defined).

H 5.2 Noether Currents *(8 points)*

Consider a classical field theory of a single scalar field $\phi(x)$ defined by the action

$$S = \int \mathrm{d}^4x \mathcal{L}(\phi, \partial_\mu \phi)\,,$$

which is invariant under (global) Lorentz transformations

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu\,.$$

(a) Show that the parameter of an infinitesimal Lorentz transformation

$$x^\mu \mapsto \left(\delta^\mu_\nu + \omega^\mu{}_\nu\right) x^\nu\,,$$

has to be antisymmetric in its upper indices, i.e. $\omega^{\mu\nu} + \omega^{\nu\mu} = 0$. (*1 point*)

(b) Show, that the conserved currents of such an infinitesimal transformation are given by

$$(\mathcal{J}^\mu)^{\rho\sigma} = x^\sigma T^{\mu\rho} - x^\rho T^{\mu\sigma}\,,$$

where $T^{\mu\nu}$ is the *canonical energy momentum tensor.* (*5 points*)

(c) What are the conserved charges? Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3x x^i T^{00} = \text{const.}.$$

(*2 points*)

H 5.3 Electromagnetism revisited *(6 points)*

Maxwell's theory of electromagnetism (without sources) can be easily written in form language as the action

$$S = \frac{1}{4} \int \mathrm{d}^4 x \; F_{\mu\nu} F^{\mu\nu}$$

where the field strength tensor F is the exterior derivative $F = \mathrm{d}A$ of the gauge field 1-form A.

(a) Show that this reproduces Maxwell's equations as given in H 2.2(a).
Hint: Vary the action with respect to A. (*3 points*)

(b) Calculate the conserved current corresponding to the global shift symmetry $A^\mu \mapsto A^\mu + a^\mu$. What is the conserved charge? (*2 points*)

(c) Show that the canonical energy momentum tensor is given by

$$T^{\mu\nu} = F^{\lambda\mu} F_\lambda{}^\nu - \frac{1}{4} \eta^{\mu\nu} F^2 - F^{\mu\kappa} \partial_\kappa A^\nu \,.$$

(*1 point*)