Physikalisches Institut
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Theoretische Physik

Exercise 6
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Exercises on General Relativity and Cosmology

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http://www.th.physik.uni-bonn.de/people/forste/exercises/ss2013/gr

–Home Exercises–

H 6.1 Charts for S^2 *(9 points)*

The 2-sphere S^2 can be embedded in 3-dimensional Euclidean space by the defining equation

$$x^2 + y^2 + z^2 = 1\,.$$

(a) Use the defining equation to construct *open* charts for S^2. Give the transition functions and check that they are smooth.
Hint: Use the square-root function. Note however, that you need more than two charts. (*2 points*)

Another example of a manifold is given by $\mathbb{C}P^1$. It is defined as the space of all lines in $\mathbb{C}^2$ that pass through the origin. Note that we here refer to complex lines, i.e. copies of $\mathbb{C}$. An element of $\mathbb{C}P^1$ is denoted by

$$[z_1 : z_2] = \{(z_1, z_2) \neq (0,0) | z_1, z_2 \in \mathbb{C}\} \,/\, \sim\,,$$

where the equivalence relation $\sim$ is given by

$$(z_1, z_2) \sim (w_1, w_2) \Leftrightarrow \exists \lambda \in \mathbb{C}^* \text{ s.t. } (z_1, z_2) = \lambda(w_1, w_2)\,.$$

Here $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

(b) Show that each element in $\mathbb{C}P^1$ can be represented as either $[1 : a]$ or as $[b : 1]$. (*1 point*)

(c) For now restrict to a real picture, i.e. $\mathbb{R}P^1$. Consider $\mathbb{R}^2$ and draw the lines $x = 1$ and $y = 1$ (Note that these lines are not elements of the projective space!). What do the representatives from (b) correspond to in this picture? How many lines (that are elements of $\mathbb{R}P^1$) are there that do not intersect $x = 1$? Which representation of $\mathbb{R}P^1$ emerges from this picture? (*3 points*)

(d) Conclude that we can endow $\mathbb{C}P^1$ with two charts both being isomorphic to $\mathbb{C}$. Show that the transition function is given by

$$\phi : \mathbb{C}^* \to \mathbb{C}^*, \qquad z_1 \mapsto z_2 = z_1^{-1}\,.$$

Here z_i denotes the coordinate on the respective copy of $\mathbb{C}$. (*1 point*)

(e) Show that $\mathbb{C}P^1$ is diffeomorphic to S^2. To do so you may use the charts that are provided by the stereographic projection.
Hint: The charts for the stereographic projection are constructed in Carroll and may be used without deriving them again. (*2 points*)

H 6.2 Homeomorphisms (*4 points*)
Show that

(a) $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. (*1 point*)

(b) S^1 is not homeomorphic to $\mathbb{R}$. (*1 point*)

(c) S^1 is not homeomorphic to S^2. (*1 point*)

(d) the open interval (a, b) is homeomorphic to $\mathbb{R}$ for any $a < b$. (*1 point*)

H 6.3 Lie Groups (*7 points*)
A Lie group G is a differentiable manifold which is endowed with a group structure such that the group operations

(i) $\cdot : G \times G \to G\,, \quad (g_1, g_2) \mapsto g_1 \cdot g_2$

(ii) ${}^{-1} : G \to G\,, \quad g \mapsto g^{-1}$

are differentiable. Usually the product symbol is omitted: $g_1 \cdot g_2$ is written as $g_1 g_2$. The dimension of a Lie group G is defined to be the dimension of G as a manifold.

(a) Show that $\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$ is a Lie group with respect to multiplication. (*1 point*)

(b) Show that $\mathbb{R}$ is a Lie group with respect to addition. (*1 point*)

(c) Show that $\mathbb{R}^2$ is a Lie group with respect to addition defined by $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$. (*1 point*)

(d) Let S^1 be the unit circle in the complex plane,

$$S^1 = \{\mathrm{e}^{\mathrm{i}\theta} | \theta \in \mathbb{R} \pmod{2\pi}\}\,.$$

Define the group operations as $\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\phi} = \mathrm{e}^{\mathrm{i}(\theta+\phi)}$ and $\left(\mathrm{e}^{\mathrm{i}\theta}\right)^{-1} = \mathrm{e}^{-\mathrm{i}\theta}$. Show that S^1 is a Lie group. (*1 point*)

(e) Show that the manifold of all the $n \times n$ real matrices g with non-vanishing determinant is a Lie group. This is the so called general linear group $\mathrm{GL}(n, R)$. Moreover, show that the manifold $\mathrm{SL}(n, R)$ of all $n \times n$ matrices with determinant 1 is a subgroub of $\mathrm{GL}(R, n)$. What are the dimensions of $\mathrm{GL}(n, R)$ and of $\mathrm{SL}(n, R)$? (*3 points*)