Physikalisches Institut
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Exercises on General Relativity and Cosmology

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–Home Exercises–

In the lecture you have already seen the definition of differentiable manifolds. For completeness and to set our notation we repeat the most important concepts here.

Definiton 1. *Let X be a set and $\mathcal{T} = \{U_i | i \in I\}$ a collection of subsets of X. The pair $(X, \mathcal{T})$ is a* **topological space** *if $\mathcal{T}$ satisfies the following requirements:*

(i) $\emptyset, X \in \mathcal{T}$,

(ii) If J is any subcollection of I, the set $\{U_j | j \in J\}$ satisfies $\bigcup_{j\in J} U_j \in \mathcal{T}$,

(iii) If K is any finite subcollection of I, the set $\{U_k | k \in K\}$ satisfies $\bigcap_{k\in K} U_k \in \mathcal{T}$.

The U_i are called **open sets** *and $\mathcal{T}$ is said to give a* **topology** *to X.*

Definiton 2. *Let M be a topological space. It is an m-dimensional* **differentiable manifold***, if*

(i) M is provided with a family of pairs $\{(U_i, \varphi_i) | i \in I\}$,

(ii) $\{U_i | i \in I\}$ is a family of open sets which covers M, ie. $\bigcup_{i\in I} U_i = M$. φ_i is a homeomorphism from U_i onto an open subset U_i' of $\mathbb{R}^m$,

(iii) given U_i and U_j such that $U_i \cap U_j \neq \emptyset$, the map $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j)$ to $\varphi_i(U_i \cap U_j)$ is infinitely differentiable.

The pair (U_i, φ_i) is called a **chart***, while the whole family $\{(U_i, \varphi_i) | i \in I\}$ is called an* **atlas***. ϕ_i is called* **coordinate (function)** *and the ψ_{ij} are called* **transition functions** *or* **coordinate transformations***. Note also that the homeomorphism φ_i is represented by m functions $\{x^1(p), \ldots, x^m(p)\}$ and the set $\{x^\mu(p)\}$ is also called coordinate (of p).*

Now tangent vectors are maps from (differentiable) functions $f : M \to \mathbb{R}$ to elements of the vector space $\mathbb{R}^m$, defined via the *directional derivative* of f along a curve at a point $p \in M$. Clearly the tangent vectors of a curve are in one-to-one correspondence to the curves along which the directional derivative is taken (up to equivalence). Formalizing this statement we arrive at

Definiton 3. *Let M be an m-dimensional differentiable manifold. Let $p \in M$ and (U, φ) a chart of M with $p \in U$. Let $\Gamma = \{c : [a,b] \to M | 0 \in [a,b] \text{ and } c(0) = p\}$ be a set of curves. The* **tangent space** *$T_p(M)$ of M at p is given by the set of equivalence classes of curves,*

$$[c] = \left\{ \tilde{c} \in \Gamma \,\middle|\, \tilde{c}(0) = c(0) \text{ and } \left.\frac{\mathrm{d}\varphi(\tilde{c}(t))}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}\varphi(c(t))}{\mathrm{d}t}\right|_{t=0} \right\} .$$

The elements X of the tangent space $T_p(M)$ are called **(tangent) vectors** *and their action on a function $f : M \to \mathbb{R}$ is given by*

$$X[f] \equiv \left.\frac{\mathrm{d}f(c(t))}{\mathrm{d}t}\right|_{t=0} \equiv X^\mu \, \frac{\partial f(\varphi^{-1}(x))}{\partial x^\mu} .$$

The X^μ are then called **components** *of the vector X.*

Remarks (without proof):

- The disjoint union of the tangent spaces to all points of the manifold $T(M) = \bigcup_{p \in M} \{(p,q) | q \in T_p(M)\}$ is a vector bundle, called the **tangent bundle**.
- We write shorthand $\frac{\partial f}{\partial x^\mu}$ for $\frac{\partial f(\varphi^{-1}(x))}{\partial x^\mu}$.
- The last equality of definition 3 implies $X^\mu = \left.\frac{\mathrm{d}x^\mu(c(t))}{\mathrm{d}t}\right|_{t=0}$.
- $T_p(M)$ is an m-dimensional vector space with basis $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$.
- Let $p \in U_i \cap U_j$ and $x = \varphi_i(p)$, $y = \varphi_j(p)$. Then we have two expressions for $X \in T_p(M)$, $X = X^\mu \frac{\partial}{\partial x^\mu} = \tilde{X}^\mu \frac{\partial}{\partial y^\mu}$, which are related by $\tilde{X}^\mu = X^\nu \frac{\partial y^\mu}{\partial x^\nu}$.

Now since $T_p(M)$ is a vector space there exists a dual to it whose elements are linear functions from $T_p(M)$ to $\mathbb{R}$:

Definiton 4. *Let M be a differentiable manifold, $p \in M$ a point and $T_p(M)$ the tangent space to p. The* **cotangent space** *$T_p^*(M)$ is the space of linear functions $T_p(M) \to \mathbb{R}$. The elements of $T_p^*(M)$ are called* **dual** *or* **cotangent vectors**.

Remarks:

- Analogous to the case of the tangent spaces, the disjoint union of the cotangent spaces to all points of the manifold $T^*(M) = \bigcup_{p \in M} \left\{(p,q) | q \in T_p^*(M)\right\}$ is a vector bundle, called the **cotangent bundle**.
- Dual vectors are differential **one-forms**.
- The simplest example of a one-form is the differential $\mathrm{d}f$ of a function $f : M \to \mathbb{R}$. The action of $\mathrm{d}f \in T_p^*(M)$ on $V \in T_p(M)$ is given by $\langle \mathrm{d}f, V \rangle \equiv V[f] = V^\mu \frac{\partial f}{\partial x^\mu}$. Noting that $\mathrm{d}f$ is expressed in terms of the coordinate $x = \varphi(p)$ as $\mathrm{d}f = \frac{\partial f}{\partial x^\mu} \mathrm{d}x^\mu$ it is natural to regard $\{\mathrm{d}x^\mu\}$ as a basis of $T_p^*(M)$. Moreover it is the dual basis, since $\left\langle \mathrm{d}x^\mu, \frac{\partial}{\partial x^\nu} \right\rangle = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\nu_\mu$.

- Writing an arbitrary one-form ω as $\omega = \omega_\mu \mathrm{d}x^\mu$, where the ω_μ are the **components** of ω, one can define the **inner product** $\langle\ ,\ \rangle : T_p^*(M) \times T_p(M) \to \mathbb{R}$ by

$$\langle \omega, V \rangle = \omega_\mu V^\nu \left\langle \mathrm{d}x^\mu, \frac{\partial}{\partial x^\nu} \right\rangle = \omega_\mu V^\mu \, .$$

- Just as for the case of vectors, let $p \in U_i \cap U_j$ and $x = \varphi_i(p)$, $y = \varphi_j(p)$. Then we have two expressions for $\omega \in T_p^*(M)$, $\omega = \omega_\mu \mathrm{d}x^\mu = \tilde{\omega}_\nu \mathrm{d}y^\nu$. From $\mathrm{d}y^\nu = \frac{\partial y^\nu}{\partial x^\mu} \mathrm{d}x^\mu$ we find that they are related by $\tilde{\omega}_\nu = \omega_\mu \frac{\partial x^\mu}{\partial y^\nu}$.

Given the existence of the tangent and cotangent space we can make use of the *tensor product* of vector spaces that we defined in H 4.1, to build (q, r) **tensors** as elements of

$$\mathfrak{T}^q_{r,p}(M) \equiv \underbrace{T_p(M) \otimes \cdots \otimes T_p(M)}_{q \text{ times}} \otimes \underbrace{T_p^*(M) \otimes \cdots \otimes T_p^*(M)}_{r \text{ times}} ,$$

which can be written in terms of the above bases as

$$\mathfrak{T}^q_{r,p}(M) \ni T = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \, \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_q} \otimes \mathrm{d}x^{\nu_1} \otimes \cdots \otimes \mathrm{d}x^{\nu_r} \, .$$

In the same way we can define **differential forms** of rank r as completely antisymmetric $(0, r)$ tensors,

$$A_r = \frac{1}{r!} A_{\nu_1 \dots \nu_r} \, \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_r} \, .$$

Of course, we can define vectors and tensors to each point on the manifold in a continuous way, which leads to

Definiton 5. *Let M be an m-dimensional differentiable manifold with tangent bundle $T(M)$. Now assign a vector $X|_p$ to each point on M,*

$$X = \left\{ X|_p \in T_p(M) | p \in M \right\} \, .$$

If for every smooth function $f : M \to \mathbb{R}$, $X(f) : M \to \mathbb{R}$ is itself a smooth function, X is called a **vector field**. *Analogously if we smoothly assign a tensor to each point of the manifold we get a* **tensor field**.

Now for a special class of manifolds, called **Riemannian manifolds** it is possible to globally define a metric:

Definiton 6. *Let M be a differentiable manifold. A* **pseudo-Riemannian metric** *g on M is a type $(0, 2)$ tensor field on M which satisfies the following axioms at each point $p \in M$*

(i) $g|_p (U, V) = g|_p (V, U)$, $\forall U, V \in T_p(M)$,

(ii) if $g|_p (U, V) = 0$ *for any* $U \in T_p(M)$, *then* $V = 0$.

If in addition, for all $U \in T_p(M)$ it satisfies $g|_p (U, U) \geq 0$, where the equaility holds only when $U = 0$, it is called a **Riemannian metric**.

Remarks:

- Let (U, φ) be a chart on M, $\{x^\mu\}$ the coordinates and $p \in U$, then g is expanded in terms of $\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ as $g\big|_p = g_{\mu\nu}(p)\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$, which exactly recovers the definition of the metric as an infinitesimal distance squared $\mathrm{d}s^2$.
- One usually omits the p in g unless it may cause confusion.
- For Riemannian manifolds the metric provides a canonical choice for the bilinear form from H 2.1.

Definiton 7. *Let M, N be differentiable manifolds and let $f : M \to N$ be a smooth map. Then this map naturally induces a map*

$$F_* : T_p(M) \to T_{f(p)}(N)\,,$$

called the **differential map** *or* **push-forward** *of f. Let $V \in T_p(M)$ and $g : N \to \mathbb{R}$ be a smooth function. Then the action of f_*V on g is defined by*

$$(f_*V)[g] \equiv V[g \circ f]\,.$$

Furthermore, f induces a map

$$f^* : T^*_{f(p)}(N) \to T^*_p(M)\,,$$

called the **pull-back** *of f. For $V \in T_p(M)$ and $\omega \in T^*_{f(p)}(N)$, the pull-back of ω by f^* is defined by*

$$\langle f^*\omega, V\rangle = \langle \omega, f_*V\rangle\ .$$

H 7.1 Coordinate Transformations *(3 points)*

How do the basis vectors of $T_p(M)$ and $T^*_p(M)$ transform under smooth and homeomorphic coordinate transformations $x^\mu \mapsto x'^\mu(x)$? How do (q, r)-tensors transform? Show that partial derivatives $\partial_\mu W_\nu$ of the components W_ν of a vector W do not transform as tensor components.

H 7.2 Pull-back and Push-forward *(6 points)*

Let M, N be differentiable manifolds, (U, φ) a chart on M, (V, ψ) a chart on N and $p \in U$. Let $f : M \to N$ be a smooth map with $f(p) \in V$. Write $x = \varphi(p)$ and $y = \psi(f(p))$.

(a) Let $T_p(M) \ni V = V^\mu \frac{\partial}{\partial x^\mu}$ and $f_*V = W^\alpha \frac{\partial}{\partial y^\alpha}$. Show that

$$W^\alpha = V^\mu \frac{\partial y^\alpha}{\partial x^\mu}\,.$$

(2 points)

(b) Show that for $\omega = \omega_\alpha \mathrm{d}y^\alpha \in T^*_{f(p)}(N)$, the induced one form $f^*\omega = \xi_\mu \mathrm{d}x^\mu \in T^*_p(M)$ has components

$$\xi_\mu = \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu}\,.$$

(2 points)

(c) Assume M to be Riemmanian. Consider a curve $\gamma : [a,b] \to M, \lambda \mapsto p(\lambda)$ and assume for simplicity that $\mathrm{Im}\,\gamma \subset M$ can be covered by a single chart. Calculate the pull-back of the metric g onto the curve. What is the geometrical meaning of this expression? *(2 points)*

H 7.3 Lie Bracket *(5 points)*

Let M be a differentiable manifold, (U, φ) a chart on M, $p \in U$, $x = \varphi(p)$. Let $X = X^\mu \frac{\partial}{\partial x^\mu}$, $Y = Y^\mu \frac{\partial}{\partial x^\mu}$, $Z = Z^\mu \frac{\partial}{\partial x^\mu}$ be vector fields on M. Then the **Lie bracket** $[X, Y]$ is defined by

$$[X, Y]f = X[Y[f]] - Y[X[f]]\,,$$

where $f : M \to \mathbb{R}$ is a smooth function.

(a) Show that

$$[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu\,,$$

where we write shorthand ∂_μ for $\frac{\partial}{\partial x^\mu}$. *(2 points)*

(b) Show that the Lie bracket

(i) is bilinear,

(ii) is skew-symmetric,

(iii) fulfills the *Jacobi identity*

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0\,.$$

(2 points)

(c) Show that $[X, Y]$ transforms as a vector field under smooth and homeomorphic coordinate transformations $x^\mu \mapsto x'^\mu(x)$. *(1 point)*

H 7.4 Explicit Calculations on Manifolds *(5 points)*

Consider $\mathbb{R}^3$ as a manifold with flat Euclidean metric and coordinates $\{x, y, z\}$.

(a) A particle moves along a parameterized curve given by

$$x(\lambda) = \cos\lambda\,, \quad y(\lambda) = \sin\lambda\,, \quad z(\lambda) = \lambda\,.$$

Express the path of the curve in spherical polar coordiantes. *(2 points)*

(b) Calculate the components of the tangent vector to the curve in both the Cartesian and spherical polar coordinate systems. *(2 points)*

Now consider *prolate spheroidal coordinates*, which can be used to simplify the Kepler problem in classical mechanics. They are given by

$$\begin{aligned} x &= \sinh\chi \sin\theta \cos\phi \\ y &= \sinh\chi \sin\theta \sin\phi \\ z &= \cosh\chi \cos\theta \,. \end{aligned}$$

Consider the plane $y = 0$.

(c) What is the coordinate transformation matrix $\frac{\partial x^\mu}{\partial x'^\nu}$ relating (x, z) to (χ, θ)? *(1 point)*

(d) What does the line element $\mathrm{d}s^2$ look like in prolate spheroidal coordinates? *(1 point)*