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Theoretische Physik

Exercise 8
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Exercises on General Relativity and Cosmology

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–Home Exercises–

Let M be a Riemannian manifold with metric g and two charts (U, ϕ), (V, ψ) which fulfill $U \cap V \neq \emptyset$. Denote the coordinates with respect to the two charts by $x = \phi(p)$ and $y = \psi(p)$, where p is any point in U or V respectively. Denote the space of vector fields on M by $\mathfrak{X}(M)$.

H 8.1 Properties of affine Connections *(8 points)*

As we have seen in the lecture, an **affine connection** ∇ is a map

$$\begin{aligned} \nabla: \quad \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (X, Y) &\mapsto \nabla_X Y \,, \end{aligned}$$

which satisfies

$$\begin{aligned} \nabla_X(Y+Z) &= \nabla_X Y + \nabla_X Z \,, \\ \nabla_{(X+Y)} Z &= \nabla_X Z + \nabla_Y Z \,, \\ \nabla_{(fX)} Y &= f \nabla_X Y \,, \\ \nabla_X(fY) &= X[f] Y + f \nabla_X Y \,, \end{aligned}$$

where $X, Y, Z \in \mathfrak{X}(M)$, and $f : M \to \mathbb{R}$ is a smooth function. The **connection components** $\Gamma^\lambda{}_{\nu\mu}$ are given by

$$\nabla_{\partial_\nu} \partial_\mu \equiv \nabla_\nu \partial_\mu = \Gamma^\lambda{}_{\nu\mu} \partial_\lambda \,.$$

Using this one finds that for $X = X^\mu \partial_\mu$, $Y = Y^\mu \partial_\mu$,

$$\nabla_X Y = X^\mu \left(\frac{\partial Y^\lambda}{\partial x^\mu} + Y^\nu \Gamma^\lambda{}_{\mu\nu} \right) \partial_\lambda \equiv X^\mu \left(\nabla_\mu Y \right)^\lambda \partial_\lambda \,.$$

Now in order to define the action of the connection on general tensor fields, one first imposes the action of ∇_X on a function $f : M \to \mathbb{R}$ to be

$$\nabla_X f = X[f]$$

and then imposes the *Leibniz rule*,

$$\nabla_X (T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2) \,,$$

where $X \in \mathfrak{X}(M)$ and T_1, T_2 are tensor fields of arbitrary types.

(a) Let $\omega = \omega_\nu \mathrm{d}x^\nu$ be a one-form field and $X = X^\mu \partial_\mu$ a vector field. Derive the action of an affine connection ∇ on ω,

$$(\nabla_X \omega)_\nu = X^\mu \partial_\mu \omega_\nu - X^\mu \Gamma^\lambda{}_{\mu\nu} \omega_\lambda \,,$$

by looking at $\nabla_X (\langle \omega, Y \rangle)$. *(2 points)*

It is easy to generalize this result to tensors of arbitrary type. Let T be a (q, r) tensor. Then

$$\begin{aligned}(\nabla_X T)^{\mu_1 \ldots \mu_q}{}_{\nu_1 \ldots \nu_r} = {} & X^\rho \partial_\rho T^{\mu_1 \ldots \mu_q}{}_{\nu_1 \ldots \nu_r} + X^\rho \Gamma^{\mu_1}{}_{\rho\kappa} T^{\kappa \mu_2 \ldots \mu_q}{}_{\nu_1 \ldots \nu_r} + \cdots + X^\rho \Gamma^{\mu_q}{}_{\rho\kappa} T^{\mu_1 \ldots \mu_{q-1} \kappa}{}_{\nu_1 \ldots \nu_r} \\ & - X^\rho \Gamma^\kappa{}_{\rho\nu_1} T^{\mu_1 \ldots \mu_q}{}_{\kappa \nu_2 \ldots \nu_r} - \cdots - X^\rho \Gamma^\kappa{}_{\rho\nu_r} T^{\mu_1 \ldots \mu_q}{}_{\nu_1 \ldots \nu_{r-1} \kappa}\end{aligned}$$

(b) Consider the region $U \cap V$. Then the affine connection ∇ has components $\tilde{\Gamma}^\gamma_{\alpha\beta}$, given by

$$\nabla_{\frac{\partial}{\partial y^\alpha}} \left(\frac{\partial}{\partial y^\beta} \right) = \tilde{\Gamma}^\gamma_{\alpha\beta} \frac{\partial}{\partial y^\gamma} \,,$$

in terms of the coordinates $y = \psi(p)$. Show that the connection components are related by (transform as)

$$\tilde{\Gamma}^\gamma_{\alpha\beta} = \frac{\partial x^\lambda}{\partial y^\alpha} \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial y^\gamma}{\partial x^\nu} \Gamma^\nu{}_{\lambda\mu} + \frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta} \frac{\partial y^\gamma}{\partial x^\nu} \,.$$

Show that this transformation rule indeed makes $\nabla_X Y$ a vector ($Y, X \in \mathfrak{X}(M)$). *(2 points)*

(c) Show further, that the components

$$(\nabla_\mu \omega)_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda{}_{\mu\nu} \omega_\lambda$$

transform as tensor components, where $\omega = \omega_\nu \mathrm{d}x^\nu$ is a one-form field. *(1 point)*

Now we demand that the metric $g_{\mu\nu}$ be *covariantly constant*, that is, if two vectors X and Y are parallel transported, then the inner product between them remains constant under parallel transport. Let V be a tangent vector to an arbitrary curve along which the vectors are parallel transported. Then we have

$$\begin{aligned} 0 = \nabla_V (g(X, Y)) &= V^\kappa \left[(\nabla_\kappa g)(X, Y) + g(\nabla_\kappa X, Y) + g(X, \nabla_\kappa Y) \right] \\ &= V^\kappa X^\mu Y^\nu (\nabla_\kappa g)_{\mu\nu} \,, \end{aligned}$$

where we have used that $V^\kappa \nabla_\kappa X = V^\kappa \nabla_\kappa Y = 0$. Since this is true for any curves and vectors, this means

$$(\nabla_\kappa g)_{\mu\nu} = 0 \,.$$

If this condition is satisfied, the connection ∇ is said to be **metric compatible**.

(d) Show that for a metric compatible connection ∇ with components $\Gamma^{\lambda}{}_{\mu\nu}$ the equation

$$\partial_\lambda g_{\mu\nu} - \Gamma^{\kappa}{}_{\lambda\mu} g_{\kappa\nu} - \Gamma^{\kappa}{}_{\lambda\nu} g_{\kappa\mu} = 0$$

holds. Show that this implies

$$\Gamma^{\kappa}{}_{(\mu\nu)} = \tilde{\Gamma}^{\kappa}_{\mu\nu} + \frac{1}{2}\left(T_{\nu}{}^{\kappa}{}_{\mu} + T_{\mu}{}^{\kappa}{}_{\nu}\right) ,$$

where $\Gamma^{\kappa}{}_{(\mu\nu)} = \frac{1}{2}\left(\Gamma^{\kappa}{}_{\mu\nu} + \Gamma^{\kappa}{}_{\nu\mu}\right)$, $T^{\kappa}{}_{\lambda\mu} = 2\Gamma^{\kappa}{}_{[\lambda\mu]} = \Gamma^{\kappa}{}_{\lambda\mu} - \Gamma^{\kappa}{}_{\mu\lambda}$ and

$$\tilde{\Gamma}^{\kappa}_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right)$$

are the **Christoffel symbols**.
Hint: Take a suitable linear combination of copies of the equation $(\nabla_\lambda g)_{\mu\nu} = 0$ *with cyclic permutations of* (λ, μ, ν). *(3 points)*

This implies, that the connection coefficients Γ are given by

$$\Gamma^{\kappa}{}_{\mu\nu} = \tilde{\Gamma}^{\kappa}_{\lambda\mu} + K^{\kappa}{}_{\mu\nu} ,$$

where

$$K^{\kappa}{}_{\mu\nu} \equiv \frac{1}{2}\left(T^{\kappa}{}_{\mu\nu} + T_{\mu}{}^{\kappa}{}_{\nu} + T_{\nu}{}^{\kappa}{}_{\mu}\right)$$

is called the **contorsion**, whereas $T^{\kappa}{}_{\mu\nu}$ is called the **torsion tensor**. This implies, that if the torsion tensor vanishes on a manifold M, the components of the metric connection ∇ are given by the Christoffel symbols. The connection is then called the **Levi-Civita connection**.

H 8.2 Geodesic equation *(8 points)*

In the lecture we have seen that a curve is a geodesic iff there is a parameterisation such that it parallel transports its own tangent vector. In the case in which the conection on the manifold is given by the Levi-Civita connection, given two points, a geodesic is also that curve c connecting the points, that extremizes the length functional

$$L(c) = \int_c \mathrm{d}s = \int_{\lambda_0}^{\lambda_1} \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda\,,$$

where λ is the parameter of the curve. (Note that for simplicity we assume that $\mathrm{Im}c \subset M$ is covered by a single chart.)

(a) By varying the above functional, derive the geodesic equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^{\mu}{}_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = \frac{1}{e}\frac{\mathrm{d}e}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} ,$$

where $e = \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}$. *(2 points)*

(b) Show that if you parameterise the curve by its proper time τ the geodesic equation is simplified to

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} = 0\,.$$

(*1 point*)

Now, as an example, let us consider geodesics of S^2 with metric $\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$.

(c) Show, that the geodesic equations take the following form

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2 = 0\,,$$
$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2} + 2\cot\theta\frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}\varphi}{\mathrm{d}s} = 0\,,$$

where s is the arc length. (*1 point*)

(d) Let $\theta = \theta(\varphi)$ be the equation of the geodesic. Show that the above equations can be written in one equation as follows

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\varphi^2} - 2\cot\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}\varphi}\right)^2 - \sin\theta\cos\theta = 0\,.$$

(*1.5 points*)

(e) Define $f(\theta) = \cot\theta$ and show that f fulfills the following differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\varphi^2} + f = 0\,.$$

What is the general solution? What do the geodesics of S^2 look like? (*2.5 points*)

H 8.3 Geometrical meaning of the torsion tensor (*4 points*)

Let $X = \epsilon^\mu\partial_\mu$ and $Y = \delta^\mu\partial_\mu$ be two infinitesimal vectors in $T_p(M)$. These vectors are regarded as small displacements and thus define two points q and s near p, whose coordinates are $x^\mu + \epsilon^\mu$ and $x^\mu + \delta^\mu$ respectively. Parallel transporting X along the line ps we obtain a vector pointing from s to some new point r_1 and similarly, by parallel transport of Y along the line pq gives a vector pointing from q to some new point r_2. (Note: In this exercise we consider a general connection ∇ with components $\Gamma^\kappa{}_{\mu\nu}$.)

(a) Show that the vectors sr_1 and qr_2 are given by

$$\epsilon^\mu - \Gamma^\mu{}_{\alpha\beta}\delta^\alpha\epsilon^\beta \quad \text{and} \quad \delta^\mu - \Gamma^\mu{}_{\beta\alpha}\delta^\alpha\epsilon^\beta\,,$$

respectively. (*2 points*)

(b) Now argue that the torsion tensor measures the failure of closure of the parallelogram made up of the small displacement vectors and their parallel transports. (*2 points*)