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Universität Bonn
Theoretische Physik

Exercise 9
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Exercises on General Relativity and Cosmology

Priv.-Doz. Dr. Stefan Förste

http://www.th.physik.uni-bonn.de/people/forste/exercises/ss2013/gr

–Home Exercises–

Let M be a (pseudo-)Riemannian manifold with metric g and two charts (U, ϕ), (V, ψ) which fulfill $U \cap V \neq \emptyset$. Denote the coordinates with respect to the two charts by $x = \phi(p)$ and $y = \psi(p)$, where p is any point in U or V respectively. Denote the space of vector fields on M by $\mathfrak{X}(M)$.

H 9.1 Symmetries and Killing vector fields *(20 points)*

General relativity is supposed to be invariant under general coordinate redefinitions. This fact is often called *diffeomorphism invariance*. This exercise studies diffeomorphisms that leave the metric unchanged, so-called **isometries**: A diffeomorphism $f : M \to M$ is an isometriy if it preserves the metric, i.e.

$$f^* g_{f(p)} = g_p \,,$$

or in components

$$\frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}(f(p)) = g_{\mu\nu}(p) \,,$$

where x and y are the coordinates of p and $f(p)$ respectively. Isometries naturally form a group which we are going to study from an infinitesimal point of view.

(a) A vector field $X = X^\mu \partial_\mu$ on M is said to be a **Killing vector field** if the small displacement $f : x^\mu \mapsto x^\mu + \epsilon X^\mu$ generates an isometry (small means $\epsilon^2 \approx 0$). Show that this is the case, if

$$X^\kappa \partial_\kappa g_{\mu\nu} + \partial_\mu X^\kappa g_{\kappa\nu} + \partial_\nu X^\kappa g_{\mu\kappa} = 0 \,.$$

These are the so-called **Killing equations**. (*1 point*)

(b) Show that the Killing equations can be written as

$$\mathcal{L}_X g_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu = 0 \,.$$

(*1 point*)

(c) Work out the Killing vector fields of the Minkowski spacetime $(\mathbb{R}^{3,1}, \eta)$ by solving the Killing equations. (*3 points*)

(d) Show that an n-dimensional Minkowski space ($n \geq 2$) is equipped with $n(n+1)/2$ Killing vector fields. Spaces which admit this number of Killing vector fields are called *maximally symmetric spaces.* *(1 point)*

(e) Verify that if the metric is independent of some coordiante x^σ the corresponding vector ∂_σ is a Killing vector. *(2 points)*

(f) Let X^μ and Y^ν be two Killing vector fields. Show that any linear combination of those two vectors is a Killing vector field and that the Lie-bracket $[X, Y]$ is a Killing vector field as well. Conclude that the set of Killing vector fields forms a Lie algebra. *(3 points)*

Now, as an example we consider the two-sphere S^2 with its usual induced metric.

(g) Write down the three Killing equations for the vector field $X = X^\theta \partial_\theta + X^\varphi \partial_\varphi$. *(1 point)*

(h) Show that X_θ is independent of θ. We may write $X_\theta(\theta, \varphi) = f(\varphi)$. By substituting this into one of the Killing equations work out that X_φ satisfies

$$X_\varphi = -F(\varphi)\sin\theta\cos\theta + g(\theta)\,,$$

where $F(\varphi)$ is the primitive of $f(\varphi)$ and $g(\theta)$ is some integration constant. *(1 point)*

(i) By plugging the result of the previous task into the last remaining Killing equation show that by separation of variables one obtains

$$\frac{\mathrm{d}g}{\mathrm{d}\theta} - 2g(\theta)\cot\theta = C\,, \tag{1}$$

$$\frac{\mathrm{d}f}{\mathrm{d}\varphi} + F(\varphi) = -C\,, \tag{2}$$

for some constant C. *(2 points)*

(j) By integrating (1) find $g(\theta)$. By differentiating (2) show that f is harmonic and write down the general solution. You should end up with

$$g(\theta) = (C_1 - C\cot\theta)\sin^2\theta\,, \tag{3}$$

$$f(\varphi) = A\sin\varphi + B\cos\varphi\,. \tag{4}$$

(2 points)

(k) Putting all results together show that a general Killing vector on S^2 is given by

$$X = A\,(\sin\varphi\,\partial_\theta + \cos\varphi\cot\theta\,\partial_\varphi) + B\,(\cos\varphi\,\partial_\theta - \sin\varphi\cot\theta\,\partial_\varphi) + C_1\,\partial_\varphi\,. \tag{5}$$

(1 point)

(l) Identify the three basis vectors of the Killing vector (5) with the angular momentum $L_i = \sum_{j,k}\epsilon_{ijk}x_j\partial_k$. Argue that the Killing vectors on S^2 generate the Lie algebra $\mathfrak{so}(3)$. Is S^2 a maximally symmetric space? *(2 points)*

H 9.2 Non-coordinate basis and vielbeins *(5 points)*

In the coordinate basis $T_p(M)$ is spanned by $\{\partial_\mu\}$ and $T_p^*(M)$ by $\{\mathrm{d}x^\mu\}$. If M is endowed with a metric $g_{\mu\nu}$ there exists an alternative choice. Consider a $\mathrm{GL}(n,\mathbb{R})$-rotation of the basis vectors ∂_μ, i.e.

$$\hat{e}_\alpha = e_\alpha{}^\mu \partial_\mu\,, \qquad (e_\alpha{}^\mu) \in \mathrm{GL}(n,\mathbb{R})\,,$$

such that $\det(e_\alpha{}^\mu) > 0$ in order to preserve the orientation of the manifold. In addition we require $\{\hat{e}_\alpha\}$ to be orthonormal with respect to $g_{\mu\nu}$, i.e.

$$g(\hat{e}_\alpha, \hat{e}_\beta) = e_\alpha{}^\mu e_\beta{}^\nu g_{\mu\nu} = \eta_{\alpha\beta}\,.$$

If the manifold is strictly Riemannian $\eta_{\alpha\beta}$ should be replaced by $\delta_{\alpha\beta}$. Denote the inverse of $e_\alpha{}^\mu$ by $e^\alpha{}_\mu$.

(a) Show that the components of a vector V in the new basis $\hat{e}_\alpha$ are related to the old components V^μ by $V^\alpha = e^\alpha{}_\mu V^\mu$. *(1 point)*

(b) Introduce the dual basis $\{\hat{\theta}^\alpha\}$ to $\{\hat{e}_\alpha\}$ by $\left\langle \hat{\theta}^\alpha, \hat{e}_\beta \right\rangle = \delta^\alpha_\beta$. Conclude that $\hat{\theta}^\alpha = e^\alpha{}_\mu\, \mathrm{d}x^\mu$. *(2 points)*

$\{\hat{e}_\alpha\}$ and $\{\hat{\theta}^\alpha\}$ are called the **non-coordiante basis** and $e^\alpha{}_\mu$ are called the **vielbeins**.

(c) Show that the metric is given by $\mathrm{d}s^2 = \eta_{\alpha\beta}\, \hat{\theta}^\alpha \otimes \hat{\theta}^\beta$. *(1 point)*

(d) Consider the standard induced metric on S^2 as in H 9.1. Calculate the non-coordinate basis $\hat{\theta}^\alpha$ as well as the *zweibeins* $e^\alpha{}_\mu$. *(1 point)*

The non-coordinate basis is of great interest in general relativity, because it allows for the definition of spinors on curved spacetimes[1].

H 9.3 Curvature and Riemann tensor *(15 points)*

As we have seen in H 8.1, the connection components $\Gamma^\mu_{\alpha\beta}$ do not transform tensorially under coordinate redefinitions. Hence once cannot expect that they have an intrinsic geometrical meaning as a measure of how much a manifold is curved. For example, on a flat space $\Gamma^\mu{}_{\alpha\beta}$ vanish for Cartesian coordinates but fail to do so in polar coordinates. Intrinsic objects that measure the curvature are the torsion tensor and the Riemann tensor. We already discussed the former one in H 8.3, so this exercise is dedicated to the Riemann tensor

$$R^\mu{}_{\alpha\beta\gamma} = \partial_\beta \Gamma^\mu{}_{\alpha\gamma} - \partial_\gamma \Gamma^\mu{}_{\alpha\beta} + \Gamma^\delta{}_{\alpha\gamma}\Gamma^\mu{}_{\delta\beta} - \Gamma^\delta{}_{\alpha\beta}\Gamma^\mu{}_{\delta\gamma}\,.$$

Note that the Riemann tensor is defined without reference to any metric and therefore the above formula holds for every connection with components $\Gamma^\mu{}_{\alpha\beta}$.

(a) Consider an infinitesimal parallelogram $pqrs$ whose coordinates are x^μ, $x^\mu + \epsilon^\mu$, $x^\mu + \epsilon^\mu + \delta^\mu$ and $x^\mu + \delta^\mu$, respectively (here we assume that p,q,r,s are all covered by the same chart (U,ϕ)). Take a vector $V_0 \in T_p(M)$, parallel transport it along

[1] The curved spacetime counterparts to the γ-matrices in flat spacetime are defined as $\gamma^\mu = e_\alpha{}^\mu \gamma^\alpha$ and fulfill $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

the curve $C = pqr$ and call the resulting vector $V_C(r) \in T_r(M)$. Similarly, parallel transport of V_0 along $C' = psr$ yields another vector $V_{C'}(r) \in T_r(M)$. Show that the difference is given by

$$V^\mu_{C'}(r) - V^\mu_C(r) = V_0^\kappa R^\mu{}_{\kappa\lambda\nu}\epsilon^\lambda\delta^\nu \,.$$

(*4 points*)

We have seen in the lecture, that it is always possible to locally find coordinates on M such that at the point p, $\partial_\rho g_{\mu\nu}\big|_p = 0$. Hence the connection components vanish at that point. These coordinates are called *locally inertial coordinates.* Note that the second derivatives of the metric do not vanish in this coordinate system![2] We will, in the following use this coordinate system to simplify some calculations. This is possible, because if one finds a purely tensorial equation, then (because of its transformation behaviour under general coordinate transformations) it is true in every coordinate system.

(b) Consider the Riemann tensor with all indices lowered, $R_{\mu\alpha\beta\gamma} = G_{\mu\kappa}R^\kappa{}_{\alpha\beta\gamma}$. Use locally inertial coordinates to deduce the symmetry properties of the curvature tensor, i.e.

$$\begin{aligned} R_{\kappa\lambda\mu\nu} &= -R_{\kappa\lambda\nu\mu}\,,\\ R_{\kappa\lambda\mu\nu} &= -R_{\lambda\kappa\mu\nu}\,,\\ R_{\kappa\lambda\mu\nu} &= R_{\mu\nu\kappa\lambda}\,. \end{aligned}$$

(*2 points*)

(c) Show that the sum of cyclic permutations of the last three indices of the curvature tensor vanishes, i.e.

$$R_{\kappa\lambda\mu\nu} + R_{\kappa\mu\nu\lambda} + R_{\kappa\nu\lambda\mu} = 0\,, \qquad 1^{\text{st}}\,\text{Bianchi identity}\,. \tag{6}$$

(*1 point*)

(d) Use the results in (b) to show that (6) is equivalent to the vanishing of the antisymmetric part of the last three indices of the Riemann tensor,

$$R_{\kappa[\mu\nu\lambda]} = 0\,.$$

(*1 point*)

(e) Given these relationships between the different components of the Riemann tensor, how many independent quantities remain? Deduce the number of independent components of the Riemann tensor in n dimensions. (*2 points*)

(f) Make use of locally inertial coordinates once more to prove

$$\nabla_{[\mu}R_{\kappa\lambda]\rho\sigma} = 0\,, \qquad 2^{\text{nd}}\,\text{Bianchi identity}\,. \tag{7}$$

(*3 points*)

[2] The metric at a point q near p can then be expanded as $g_{\mu\nu}(q) = \eta_{\mu\nu} + \frac{1}{3}R_{\mu\lambda\nu\rho}q^\lambda q^\rho + \ldots$. Note that in this coordinate system p has coordinates $x = (0,\ldots,0)$.

(g) By contracting indices of the second Bianchi identity (7) twice, show that

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2} \nabla_\nu R \,.$$

(2 points)