Exercises on Advanced Topics in String Theory

Priv.-Doz. Dr. Stefan Förste

http://www.th.physik.uni-bonn.de/people/forste/exercises/strings15

–Class Exercises–

C 0.1 Euler number of S^2 and D^2

In this exercise we want to discuss two methods of calculating the Euler number χ. The Euler number is a topological invariant and therefore the same for spaces which are homeomorphic to each other. In string pertubation theory the Euler number is related to the order of the loop correction of string interactions. One way to calculate χ is the following: Let X be a subset of $\mathbb{R}^3$, which is homeomorphic to a polyhedron K. Then the Euler number $\chi(X)$ of X is defined by

$$\chi(X) = (\#\,\text{of vertices in } K) - (\#\,\text{of edges in } K) + (\#\,\text{of faces in } K) \tag{1}$$

(a) Calculate χ for the 2-sphere S^2 and the disk D^2 embedded in $\mathbb{R}^3$ by finding polyhedrons homeomorphic to S^2 and D^2.

An alternative definition of the Euler number for a Riemann surface Σ is given by the Gauss-Bonnet theorem

$$\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma \sqrt{h}\mathcal{R}\mathrm{d}^2\sigma + \frac{1}{2\pi}\int_{\partial\Sigma} k\mathrm{d}s. \tag{2}$$

$\mathcal{R}$ denotes the Ricci scalar of Σ and is the determinant of the metric $h = \det(h_{\alpha\beta})$ given in terms of the coordinates σ_1, σ_2 on Σ. k is the curvature along the geodesic s on the boundary of the surface $\partial\Sigma$

(b) Find a parametrisation for S^2 and D^2 in $\mathbb{R}^3$ and calculate the metric $g_{\mu\nu}$ and Christoffel symbols $\Gamma^\kappa{}_{\mu\nu} = \frac{1}{2}g^{\kappa\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$ for the manifolds.

(c) Calculate $\mathcal{R}$ for S^2 and D^2.
Hint: $\mathcal{R} = g^{\mu\nu}R_{\mu\nu}$ *and* $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ *with* $R^\kappa{}_{\lambda\mu\nu} = \partial_\mu\Gamma^\kappa{}_{\nu\lambda} - \partial_\nu\Gamma^\kappa{}_{\mu\lambda} + \Gamma^\eta{}_{\nu\lambda}\Gamma^\kappa{}_{\mu\eta} - \Gamma^\eta{}_{\mu\lambda}\Gamma^\kappa{}_{\nu\eta}$

(d) What is the boundary for S^2 and D^2? Calculate k for the geodesics on S^2 and D^2 and find the Euler numbers using the above results and the Gauss-Bonnet Theorem.
hint: If the geodesic $\vec{r}(s)$ *is parametrised by the arc length parameter* s *the curvature is defined by* $k = \|\frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\|$, *with* $\vec{T}$ *the unit Tangentvector on* $\vec{r}(s)$.

–Home Exercises– Due to: 29.04.2015

H 1.1 $SL(2,\mathbb{Z})$ transformations and transformation properties of the torus partition function *(20 points)*

In string pertubation theory the one loop graph for closed strings has the topology of a two dimensional torus T^2. A torus can be constructed by modding out a two dimensional lattice Λ_2 out of $\mathbb{C}$. This means that points in $\mathbb{C}$ which differ by $\lambda \in \Lambda_2$ are identified

$$\mathbb{C} \ni z \sim z + \lambda. \tag{3}$$

For a given lattice $\Lambda_2 = \{\lambda = n_1\ell + n_2\tau\ell | n_1, n_2 \in \mathbb{Z}\}$ with lattice vectors ℓ and $\tau\ell$, we can specify the torus by its moduls $\tau \in \mathbb{C}$. However $SL(2,\mathbb{Z})$ transformations on τ describe the same torus.

The group $SL(2,\mathbb{Z})$ is the set of matrices given by

$$SL(2,\mathbb{Z}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z},\, ad - bc = 1 \right\}. \tag{4}$$

They act on $z \in \mathbb{C}$ by $gz = \frac{az+b}{cz+d}$. The generators of $SL(2,\mathbb{Z})$ are given by $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We further define $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm \mathbb{1}\}$ and the upper half-plane $\mathfrak{h} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$.

(a) How does T and S act on $\mathfrak{h}$? Why is it enough to consider $PSL(2,\mathbb{Z})$? *(2 points)*

(b) Show that there exists a $g_0 \in SL(2,\mathbb{Z})$ such that $\mathrm{Im}(gz) \leq \mathrm{Im}(g_0 z)$ for all $g \in SL(2,\mathbb{Z})$ and fixed $z \in \mathfrak{h}$. *(2,5 points)*

(c) Show that $|g_0 z| \geq 1$. *(1,5 points)*
Hint: Apply an S transformation on $g_0 z$

(d) Show that $|T^n g_0 z| \geq 1$ for any $n \in \mathbb{Z}$ and that one can use T transformations to achieve $-\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}$. What is the fundamental domain $\mathcal{F}$ of $SL(2,\mathbb{Z})$? *(4 points)*

(e) Argue that two moduli τ and τ' differing by $SL(2,\mathbb{Z})$ transformations describe the same torus. *(2 points)*

The torus partition function $A_0^{g=1}$ describes the one loop vacuum amplitude of closed strings. It is given by

$$A_0^{g=1} = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{4(\mathrm{Im}(\tau))^2} Z(\tau, \overline{\tau}), \tag{5}$$

where

$$Z(\tau, \overline{\tau}) = \frac{V_{26}}{\ell_s^{26}} \frac{1}{(\mathrm{Im}(\tau))^{12}} |\eta(\tau)|^{-48}, \quad \text{with} \quad \eta(\tau) = \mathrm{e}^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - \mathrm{e}^{2\pi i n \tau}) \tag{6}$$

and $\tau \in \mathbb{C}$ is the moduls of the T^2 such that

$$\mathbb{C} \ni z \sim z + 1 \quad \text{and} \quad z \sim z + \tau. \tag{7}$$

(f) Show $\mathrm{Im}(\tau)$ is the area of the T^2 with moduli τ. How does the measure $\frac{\mathrm{d}^2\tau}{4(\mathrm{Im}(\tau))^2}$ transform under $SL(2,\mathbb{Z})$? *(3 points)*

(g) Show the transformation properties of $\eta(\tau)$ under the action of the generators S and T of $SL(2,\mathbb{Z})$:

$$\eta(\tau+1) = \mathrm{e}^{\pi i/12}\eta(\tau) \quad \text{and} \quad \eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau) \tag{8}$$

and use the result to show that $A_0^{g=1}$ is invariant under $SL(2,\mathbb{Z})$. *(5 points)*

The torus partition function is *modular invariant* due to its transformation properties under the modular group. Modular invariance of closed string amplitudes can be used to uncover inconsistencies of string theories. For example we will later see, that modular invariance implies space time supersymmetry for the super string.