Physikalisches Institut
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Theoretische Physik

Exercise 2
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Exercises on Advanced Topics in String Theory

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http://www.th.physik.uni-bonn.de/people/forste/exercises/strings15

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H 2.1 Reduction to moduli of the string partition function *(16,5 points)*

The string partition function for a worldsheet with the topology of a compact Riemann surface with genus h is given by

$$Z = \mathcal{N} \int \mathcal{D}g_{ab}\mathcal{D}X^{\mu} \exp\left(- \int \mathrm{d}^2\sigma \sqrt{g} (\frac{1}{2} g^{ab}\partial_a X^{\mu} \partial_b X_{\mu} + \frac{1}{4\pi}\lambda \mathcal{R} \right), \tag{1}$$

with $\mathcal{N}$ some normalisation, σ^1, σ^2 coordinates on the worldsheet, $X^{\mu}(\sigma^1, \sigma^2)$ the mapping from the worldsheet to the target space, λ the string coupling, $\mathcal{R}$ the Ricci scalar of the worldsheet, $g = \det g_{ab}$ and g_{ab} a metric on the wordsheet.

(a) Show that the partition function in (1) is equivalent to *(0,5 points)*

$$Z = \mathcal{N} \mathrm{e}^{\lambda(2-2h)} \int \mathcal{D}g_{ab}\mathcal{D}X^{\mu} \exp\left(- \int \mathrm{d}^2\sigma \sqrt{g} (\frac{1}{2} g^{ab}\partial_a X^{\mu} \partial_b X_{\mu} \right). \tag{2}$$

The integral in (1) is highly divergent because one integrates infinitely many times over conformaly equivalent surfaces. We need to extract the divergent part and only integrate over physically inequivalent metrics. Let $\mathcal{G}_h$ denote the space of admissible metrics on a compact Riemann surface with genus h. Then

$$\langle \delta g^{(1)}_{ab}, \delta g^{(2)}_{ab} \rangle = \int \mathrm{d}^2\sigma \sqrt{g} g^{ac} g^{bd} \delta g^{(1)}_{ab} \delta g^{(2)}_{cd}. \tag{3}$$

defines a scalar $\langle \cdot, \cdot \rangle$ product of two infinitesimal variations $\delta g^{(1)}_{ab}$ and $\delta g^{(2)}_{ab}$ of the metric g_{ab}. More precise $\delta g^{(1)}_{ab}$ and $\delta g^{(2)}_{ab}$ are elements of the tangent space $T_g(\mathcal{G}_h)$ at the the point $g_{ab} \in \mathcal{G}_h$.

(b) Recall that Weyl transformations and diffeomorphisms of the metric do not change the physical results and are therefore symmetries of the worldsheet. Show that under the combined action of

- a Weyl scaling $g_{ab} \to \mathrm{e}^{\phi} g_{ab}$
- and diffeomorphism generated by a vector field $\vec{v}$

the infinitesimal variation of g_{ab} is given by

$$\delta g_{ab} = \delta\phi g_{ab} + \nabla_a v_b + \nabla_b v_a. \tag{4}$$

Split δg_{ab} into a trace part $\delta g_{ab}^{\mathrm{W}}$ and a symmetric traceless part $\delta g_{ab}^{\mathrm{D}}$ and show that the the norm of δg_{ab} is given by

$$||\delta g||^2 = ||\delta g^{\mathrm{W}}||^2 + ||\delta g^{\mathrm{D}}||^2. \tag{5}$$

How do $\delta g_{ab}^{\mathrm{W}}$ and $\delta g_{ab}^{\mathrm{D}}$ look like explicitly? (*3 points*)
Hint: Define an operator P which maps vectors $\vec{u} \in T_g(\mathcal{G}_h)$ to second-rank symmetric traceless tensors $u_a \to (P\vec{u})_{ab}$.

(c) Conformal killing vectors (CKV) $\vec{v}_{CKV}$ fulfill the condition

$$\nabla_a v_b + \nabla_b v_a = \lambda g_{ab}, \quad \text{with} \quad \lambda \in \mathbb{R}. \tag{6}$$

Show that CKV are elements $Ker(P)$ of and can be replaced by Weyl scalings. (*1 point*)

We are interested in the subspace $\mathcal{M}_h \subset \mathcal{G}_h$, containing all conformal equivalence classes of metrics, called the *moduli space*. Denoting the set of Weyl scalings as $Weyl$ and the set of diffeomorphisms as $Diff$, the moduli space is represented as

$$\mathcal{M}_h \sim \frac{\mathcal{G}_h}{Weyl \times Diff}. \tag{7}$$

In general the infinitesimal variation of a metric $g(t_i) \in \mathcal{G}_h$ is given by

$$\delta g_{ab} = \delta g_{ab}^{\mathrm{W}} + \delta g_{ab}^{\mathrm{D}} + \delta t_i \frac{\partial}{\partial t_i} g_{ab}, \tag{8}$$

where t_i are the moduli parameter.

(d) Split the term $\delta t_i \frac{\partial}{\partial t_i} g_{ab}$ into a trace and traceless symmetric part and define an operator T_{ab}^i which acts on δt_i and maps it to the traceless symmetric term of $\delta t_i \frac{\partial}{\partial t_i} g_{ab}$. (*2 points*)

In order to integrate in (1) only over physically inequivalent metrics we need to find an appropriate gauge slice in $\mathcal{G}_h$. Therefore we first need to find a slice $\tilde{\mathcal{G}}_h$ which contains all equivalence classes of metrics related by Weyl transformations. Then the gauge slice lies in $\tilde{\mathcal{G}}_h$ and is chosen in such a way that a transformation $\exp(P\vec{v})$ on a point $\tilde{g}_{ab} \in \{\text{gauge slice}\}$ leads to a point $\hat{g}_{ab}$ still in $\tilde{\mathcal{G}}_h$ but no longer in the gauge slice.

(e) Consider a point $\tilde{g}_{ab} \in \tilde{\mathcal{G}}_h$. Is it possible to act on $\tilde{g}_{ab}$ with an element $\in Diff$ in such a way, that on leaves the slice $\tilde{\mathcal{G}}_h$? Explain why! (*1 point*)

(f) Let us denote an infinitesimal variation changing the conformal equivalence class by $\delta g_{ab}^{\perp}$. It is therefore a tangent vector in the tangent space of $\mathcal{M}_h$. Why must $\delta g_{ab}^{\perp}$ be traceless? Show that $\delta g_{ab}^{\perp} \in Ker(P^{\dagger})$. *Hint: How is the angle between the tangent vectors $(P\vec{v})_{ab}$ and $\delta g_{ab}^{\perp}$?* (*1,5 points*)

(g) Let ψ^α_{ab}, $\alpha = 1, ..., \dim Ker(P^\dagger)$ be an orthonormal basis for $Ker(P^\dagger)$ and decompose $T^i_{ab}\delta t_i$ into a linear combination of basis vectors of $Ker(P^\dagger)$ and vectors of $Range(P)$. You should arrive at

$$T^i_{ab}\delta t_i = \langle \psi^\alpha, T^i\rangle \psi^\alpha_{ab}\delta t_i + \frac{\langle P\vec{v}, T^i\rangle}{||P\vec{v}||^2}(P\vec{v})_{ab}\delta t_i. \tag{9}$$

Show that the norm of δg_{ab} is given by

$$||\delta g||^2 = ||\delta\tilde{\phi}||^2 + ||P\tilde{v}||^2 + \langle \psi^\alpha, T^i\rangle\langle \psi^\alpha, T^j\rangle \delta t_i \delta t_j, \tag{10}$$

with

$$\delta\tilde{\phi} = \delta\phi + \nabla_c v^c + \frac{1}{2}\left(g^{cd}\delta t_i \frac{\partial}{\partial t_i} g_{cd}\right) \quad \text{and} \quad \tilde{v} = \left(1 + \frac{\langle P\vec{v}, T^i\delta t_i\rangle}{P\vec{v}, P\vec{v}}\right)\vec{v}. \tag{11}$$

(*2 points*)

In order to change the path integral variables from g_{ab} to ϕ, $\vec{v}$ and t_i we use the relation

$$\begin{aligned} 1 &= \int \mathcal{D}g_{ab}\exp(-||\delta g||^2/2) & (12)\\ &= J\int \mathcal{D}\phi\mathcal{D}v'^a \mathrm{d}t^1...\mathrm{d}t^n \exp\left(-[||\delta\tilde{\phi}||^2 + ||P\tilde{v}'||^2 + \langle \psi^\alpha, T^i\rangle\langle \psi^\alpha, T^j\rangle\delta t_i\delta t_j]/2\right) \end{aligned}$$

to calculate the Jacobian J. Notice that $\vec{v}'$ denotes elements from $Range(P)$. Since elements from $Ker(P)$ are orthogonal to $\vec{v}'$ we can decompose the volume of the diffeomorphism group V_{Diff} into $V^\perp_{Diff} \times V^{CKV}_{Diff}$. Let χ_i, $i = 1, ..., \dim Ker(P)$ be a basis for $Ker(P)$, then one can show that

$$V^\perp_{Diff} = V_{Diff}\left(\det\langle \chi_i, \chi_j\rangle\right)^{-1/2}. \tag{13}$$

From (12) one can show that the Jacobian for the path integral should be given by $J = \det^{1/2}(P^\dagger P)\frac{\det\langle \psi^i, T^j\rangle}{\det\langle \psi^i, \psi^j\rangle}$.

(h) Show that

$$\int \mathcal{D}g_{ab} \to V_{Diff}\int \mathcal{D}\phi \mathrm{d}t^1...\mathrm{d}t^n \left(\frac{\det(P^\dagger P)}{\det\langle \chi^i, \chi^j\rangle}\right)^{1/2}\frac{\det\langle \psi^i, T^j\rangle}{\det\langle \psi^i, \psi^j\rangle} \tag{14}$$

(*0,5 points*)

(i) Express the number of real moduli n by the genus h for a compact Riemann surface with no crosscaps and $h \geq 2$. *Hint: There are no CKV for compact Riemann surfaces with $h \geq 2$.* (*1 point*)

In the critical dimension ($D = 26$ for the bosonic string) the integrand becomes independent from ϕ and the integral $\int \mathcal{D}\phi = V_{Conf}$ can be absorbed into the normalization. It can be shown that the integral over the mappings X^μ is given by

$$\int \mathcal{D}X^\mu \exp\left(-\int \mathrm{d}^2\sigma\sqrt{g}(\frac{1}{2}g^{ab}\partial_a X^\mu \partial_b X_\mu\right) = \mathcal{V}\left(\frac{\int \mathrm{d}^2\sigma\sqrt{g}}{2\pi}\right)^{13}(\det\Delta_g)^{-13}, \tag{15}$$

with $\mathcal{V}$ the volume of space time and $\Delta_g = -\frac{1}{\sqrt{g}}\partial_a\sqrt{g}g^{ab}\partial_b$. Putting the previous results together we find that the partition function in (1) can be expressed by

$$Z = \mathcal{V}e^{\lambda(2-2h)} \int_{\mathcal{M}_h} dt^1...dt^n \left(\frac{\det(P^\dagger P)}{\det\langle\chi^i,\chi^j\rangle}\right)^{1/2} \frac{\det\langle\psi^i,T^j\rangle}{\det^{1/2}\langle\psi^i,\psi^j\rangle}\left(\frac{2\pi}{\int d^2\sigma\sqrt{g}}\det\Delta_g\right)^{-13} \tag{16}$$

(h) Now that we have the general expression for the partition function of a compact Riemann surface let us apply the results to the $h = 1$ case. The worldsheet hast the topology of a torus and Z is the torus partition function.

(i) Show that $\chi^1 = (1,0)^{\mathrm{T}}$ and $\chi^2 = (0,1)^{\mathrm{T}}$ are a possible choice for $Ker(P)$. *(0,5 points)*

(ii) Argue that $n = 2$ and show that T^i_{ab} are given by

$$T^1_{ab} = \begin{pmatrix} -\tau_1 & 1-\tau_1 \\ 1-\tau_1 & \tau_1 \end{pmatrix} \quad \text{and} \quad T^2_{ab} = \begin{pmatrix} -\tau_2 & -\tau_2 \\ -\tau_2 & \tau_2 \end{pmatrix}, \tag{17}$$

where τ_1 and τ_2 are the moduli of the torus with the $g_{ab} = |d\sigma^1 + (\tau_1 + i\tau_2)d\sigma^2|^2$. Why do T^1_{ab}, T^2_{ab} form a possible basis for $Ker(P^\dagger)$. *Hint: Since the metric is flat* $(P^\dagger T^i)_b = -2\partial^a T^i_{ab}$. *(1,5 points)*

(iii) Next calculate $\det\langle\chi^i,\chi^j\rangle$ and $\frac{\det\langle\psi^i,T^j\rangle}{\det^{1/2}\langle\psi^i,\psi^j\rangle}$ and show that $\det(P^\dagger P) = (\det(2\Delta_g))^2$. *Hint: First show* $(P^\dagger P)_{ab}v^b = 2\delta_{ab}\Delta_g v^b$ *(1 point)*

(iv) Use $\det(2\Delta_g) = \frac{1}{2}\det(2)\det(\Delta_g)$ and compute Z. You should arrive at

$$Z = \mathcal{V}\int_{\mathcal{M}_1} d^2\tau \frac{\tau_2^{10}}{(2\pi)^{13}} (\det(\Delta_g))^{-12} \det(2). \tag{18}$$

(1 point)

$\det(2)$ can be absorbed into a counterterm by modifying the action. The computation of $\det(\Delta_g)$ would lead to

$$\det\Delta_g = \tau_2^2 e^{-\pi\tau_2/3}\left|\prod_{n=1}^{\infty} 1 - e^{2i\pi n\tau}\right|^4. \tag{19}$$

plugging it into Z we arrive at the final result for the torus partition function

$$\begin{aligned} Z &= \int_{\mathcal{M}_1} \frac{d^2\tau}{2\pi\tau_2^2}(2\pi\tau_2)^{-12}e^{4\pi\tau_2}\left|\prod_{n=1}^{\infty} 1 - e^{2i\pi n\tau}\right|^{-48} \\ &= \frac{1}{2}\int_{\mathcal{F}_{\mathrm{PSL}(2,\mathbb{Z})}} \frac{d^2\tau}{2\pi\tau_2^2}(2\pi\tau_2)^{-12}e^{4\pi\tau_2}\left|\prod_{n=1}^{\infty} 1 - e^{2i\pi n\tau}\right|^{-48}, \end{aligned} \tag{20}$$

where $\mathcal{F}_{\mathrm{PSL}(2,\mathbb{Z})}$ is the fundamental domain of the modular group $PSL(2,\mathbb{Z})$. Notice that integrating over $\mathcal{F}_{\mathrm{PSL}(2,\mathbb{Z})}$ leaves an unfixed residual gauge freedom given by the diffeomorphism $\sigma^1 \to -\sigma^1$, $\sigma^2 \to -\sigma^2$. Therefore a factor of $1/2$ is necessary to remove the over-counting.

H 2.2 Physical interpretation of the torus partition function *(3,5 points)*

The effective action for a massive free scalar field ϕ is defined by

$$\mathrm{e}^{\Gamma} = \int \mathcal{D}\phi \mathrm{e}^{-S[\phi]}, \quad \text{with} \quad S[\phi] = \frac{1}{2}\int \mathrm{d}^d x \phi(-\Box + m^2)\phi. \tag{21}$$

The exponent Γ is proportional to the vacuum energy E_{vac}

$$\Gamma = \mathcal{V} E_{\text{vac}}, \tag{22}$$

where $\mathcal{V}$ is the volume of spacetime.

(a) Compute e^{Γ}. *Hint: Result is* $\mathrm{e}^{\Gamma} = (\det(-\Box + m^2))^{-1/2}$ *(2 points)*

(b) Show that

$$E_{\text{vac}} = -\frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \log(k^2 + m^2), \tag{23}$$

where k^μ is the momentum in the μ-th direction. *Hint: Use* $\mathbb{1} = \int \mathrm{d}^d k |k\rangle\langle k|$. *(1 point)*

(c) In string theory a string has many excitations generating many massive fields. Therefore the vacuum energy for a bosonic string must be the sum over all excitations

$$E_{\text{vac}} = -\frac{1}{2}\sum_i \int \frac{\mathrm{d}^d k}{(2\pi)^d} \log(k^2 + m_i^2). \tag{24}$$

Use the properties $\log A = -\lim_{\epsilon\to 0} \frac{\mathrm{d}}{\mathrm{d}\epsilon} A^{-\epsilon} = -\lim_{\epsilon\to 0} \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[\epsilon \int \frac{\mathrm{d}t}{t^{1-\epsilon}} \mathrm{e}^{-2\pi t A}\right]$ to show that the vacuum energy can be writen as a trace over the Hilbert space $\mathcal{H}$

$$E_{\text{vac}} \propto \int \frac{\mathrm{d}t}{t} \mathrm{Tr}_{\mathcal{H}}\left[\mathrm{e}^{\,\cdots}\right]. \tag{25}$$

(0,5 points)