Physikalisches Institut
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Theoretische Physik

Exercise 8
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Exercises on Advanced Topics in String Theory

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http://www.th.physik.uni-bonn.de/people/forste/exercises/strings15

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H 8.1 Basics of Jacobi & modular forms *(20 points)*

For the analysis of modularity of the partition function $Z(\tau)$ of a given CFT it is convenient to introduce the notion of modular forms and Jacobi forms that have a specific behaviour under modular transformations $\tau \mapsto \gamma(\tau)$. Here we will consider some explicit examples to compare to the mathematical definitions:
A Jacobi form of weight $k \in \mathbb{N}_0$ and index $m \in \mathbb{N}_0$ is an analytic function $f(\tau,\nu)$: $\mathfrak{h}\times\mathbb{C} \to \mathbb{C}$ that for $\gamma(\tau) = \frac{a\tau+b}{c\tau+d}$, $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, and $(\lambda,\mu) \in \mathbb{Z}^2$ satisfy

$$\begin{aligned} f(\gamma(\tau), \nu/(c\tau+d)) &= (c\tau+d)^k \exp\left(2\pi i m c\nu^2/(c\tau+d)\right) f(\tau,\nu), & (1)\\ f(\tau, \nu+\lambda\tau+\mu) &= \exp\left(2\pi i m c(-\lambda^2\tau - 2\lambda\nu)\right) f(\tau,\nu). & (2)\end{aligned}$$

Additionally, one demands at most polynomial growth of $f(\tau,\nu)$ at the cusp, i.e. at $\tau \to i\infty$. Then, this is equivalent to the existence of a Fourier expansion given by

$$f(\tau,\nu) = \sum_{n=0}^{\infty} \sum_{r\in\mathbb{Z}} c(n,m) q^n z^r \tag{3}$$

for $4nm - r^2 \geq 0$ and $q = \mathrm{e}^{2\pi i\tau}$, $z = \mathrm{e}^{2\pi i\nu}$. A modular form of weight k is a holomorphic function $f(\tau)$ on $\mathfrak{h}$ fulfilling (1) for $m = 0$.
As a first example we define the basic theta function given by its Fourier expansion

$$\theta(\nu,\tau) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i n^2\tau + 2\pi i n\nu\right) = \sum_n q^{n^2/2} z^n. \tag{4}$$

(a) Analyze its periodicity for $\nu \mapsto \nu+1$, $\nu \mapsto \nu+\tau$. *(2 points)*

(b) Prove the Poisson resummation formula for f and its Fourier transform $\hat{f}$,

$$\sum_{n=-\infty}^{\infty} f(x+nT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \hat{f}(k/T) \mathrm{e}^{2\pi i k/T x}. \tag{5}$$

(2 points)

(c) Apply Poisson resummation to analyze the modularity properties of (4),

$$\theta(\nu,\tau+1) = \theta(\nu+1/2,\tau), \quad \theta(\nu/\tau,-1/\tau) = (-i\tau)^{1/2} \mathrm{e}^{\pi i \nu^2} \theta(\nu,\tau). \tag{6}$$

(2 points)

(d) Rewrite $\theta(\nu,\tau)$ as an infinite product using Jacobi's triple product identity,

$$\prod_{n=1}^{\infty} (1-q^n)(1+q^{n-1/2})(1+q^{n-1/2}/t) = \sum_{n\in\mathbb{Z}} q^{n^2/2} t^n. \tag{7}$$

What is the bahaviour for $q \to 0$, $q \to 1$ and $z \to -q^{1/2}$? *Hint: Use $\tau \mapsto -1/\tau$ to analyze the limit $q \to 1$!* *(2 points)*

The theta function with characteristics is defined by

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (\nu,\tau) = \exp[\pi i a^2 \tau + 2\pi i a(\nu+b)]\theta(\nu+a\tau+b,\tau), \tag{8}$$

that are also denoted by

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\nu,\tau) = \theta_{00}(\nu,\tau) = \theta_3(\nu|\tau), \quad \theta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (\nu,\tau) = \theta_{01}(\nu,\tau) = \theta_4(\nu|\tau), \tag{9}$$
$$\theta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (\nu,\tau) = \theta_{10}(\nu,\tau) = \theta_2(\nu|\tau), \quad \theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (\nu,\tau) = \theta_{11}(\nu,\tau) = \theta_1(\nu|\tau).$$

(e) Determine the sum as well as product representation of (9). *(2 points)*

(f) What are the periodicity properties for $\nu \mapsto \nu+1$ and $\nu \mapsto \nu+\tau$? *(2 points)*

(g) Analyze the modularity behaviour $(\nu,\tau) \mapsto (\nu,\tau+1)$, $(\nu,\tau) \mapsto (\nu,-1/\tau)$. *(2 points)*

(h) Evaluate the theta function at $\nu = 0$ and define

$$\theta_i(\tau) \equiv \theta_i(0|\tau). \tag{10}$$

Deduce the behaviour under modular transformations. *(2 points)*

Dedekind's eta-function is defined as

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n). \tag{11}$$

(i) Show the relation

$$\eta^3(\tau) = \frac{1}{2}\theta_2(\tau)\theta_3(\tau)\theta_4(\tau) \tag{12}$$

and infer $\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau)$ as well as $\eta(\tau+1) = \exp(i\tau/12)\eta(\tau)$. *(2 points)*

(j) Finally, compare all encountered examples to (3) to read of k and m. *(2 points)*