Physikalisches Institut
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Theoretische Physik

Exercise Sheet 10
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Superstring Theory

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–Homeworks–

10.1 Spinors in various dimensions

Representations of the Clifford algebra in various dimensions are needed in supersymmetric string theories. Therefore, we investigate the construction of such representations in more detail.
A representation of the Clifford algebra in d dimensions is given by the Dirac matrices γ^μ for $\mu = 0, 1, \ldots d-1$, which satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \ .$$

In Euclidean signature we have $(\gamma^\mu)^\dagger = \gamma^\mu$ and in Minkowskian signature we have $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.
The chirality operator is defined by

$$\gamma_{d+1} = \alpha\gamma^0 \cdots \gamma^{d-1} \ .$$

The charge conjugation operator $C_\pm$ is defined by

$$(\gamma^\mu)^T = \mp C_\pm \gamma^\mu C_\pm^{-1} \ .$$

Both $C_\pm^T$ exist in even dimensions and a least one of them exists in odd dimensions. In particular, one has for $d = 2n$ and $n \in \mathbb{Z}$

$$C_\pm^T = (-1)^{\frac{1}{2}n(n\pm 1)} C_\pm$$

and for $d = 2n + 1$

$$C = \begin{cases} C_+ & \text{for } n \text{ odd} \ , \\ C_- & \text{for } n \text{ even} \ . \end{cases}$$

Schur's lemma states that an operator which commutes with all elements of an irreducible representation must be a multiple of the identity element.
In addition, we introduce the notion of antisymmetrized products of matrices

$$\gamma^{\mu_1 \cdots \mu_p} = \frac{1}{p!} \left(\gamma^{\mu_1} \cdots \gamma^{\mu_p} \pm \text{permutations} \right) \ ,$$

where the $+$ is taken if the number of permutations is even and $-$ is taken if the number of permutations is odd. The matrices $\pm\mathbb{1}, \pm\gamma^\mu, \gamma^{\mu\nu}, \ldots$ generate a finite group.

a) Argue that for odd dimensions $\gamma_{d+1} \propto \mathbb{1}$. *(1 Point)*

b) For d even determine α such that $\gamma_{d+1}^2 = 1$ and show that $(\gamma_{d+1})^\dagger = \gamma_{d+1}$. (*1 Point*)

c) Use Schur's lemme to show that $C^T = \pm C$. (*1 Point*)

d) Show that the matrices $\Sigma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$ satisfy the SO(d) or the SO($d-1,1$) algebra

$$i[\Sigma^{\mu\nu}, \Sigma^{\sigma\rho}] = \eta^{\nu\sigma}\Sigma^{\mu\rho} + \eta^{\mu\rho}\Sigma^{\nu\sigma} - \eta^{\nu\rho}\Sigma^{\mu\sigma} - \eta^{\mu\sigma}\Sigma^{\nu\rho} \ .$$

A representation for which there is a matris R such that

$$R\Sigma^{\mu\nu}R^{-1} = -\left(\Sigma^{\mu\nu}\right)^*$$

is called (pseudo) real. It is complex otherwise. In particular, one can show that $R^T = \pm R$. If a representation has a ppsitive sign, it is called *real*, whereas a representation with a negative sign is called *pseudoreal*.

e) In the $d = 2n$ even-dimensional Euclidean case, show that

$$\left(\Sigma_\pm^{\mu\nu}\right)^* = \begin{cases} -(C_\pm)\Sigma_\mp^{\mu\nu}(C_\pm)^{-1} & \text{for } n \text{ odd} \ , \\ -(C_\pm)\Sigma_\pm^{\mu\nu}(C_\pm)^{-1} & \text{for } n \text{ even} \ , \end{cases}$$

where $\Sigma_\pm^{\mu\nu} = \frac{1}{2}\Sigma^{\mu\nu}(1 \pm \gamma_{d+1})$ are generators associated to the respective chiral subspaces. (*1 Point*)

f) Evaluate for which dimensions the representations of part d) are real, pseudoreal and complex. Your result should depend on the dimension n mod 4. (*1 Point*)

g) Continuing with the cases of part d). We define

$$b_i^\pm = \frac{1}{2}\left(\gamma^{2i} \pm i\gamma^{2i+1}\right) \ , \quad i = 0, \dots, n-1 \ .$$

Show that

$$\left(b_i^\pm\right)^\dagger = b_i^\mp \ , \quad \{b_i^\pm, b_j^\mp\} = \delta_{ij} \quad \text{and} \quad \{b_i^+, b_j^+\} = \{b_i^-, b_j^-\} = 0 \ .$$

(*1 Point*)

h) Let the hightest weight state $|\Omega\rangle$ be defined by $b_i^+|\Omega\rangle = 0$ such that $|\Omega\rangle = |\frac{1}{2}, \dots, \frac{1}{2}\rangle$. All other states are given by $|\pm\frac{1}{2}, \dots, \pm\frac{1}{2}\rangle$. Show that this representation is reducible and decomposes into irreducible representations given by positive and negative chirality spinors. (*1 Point*)

i) Show that $d = 8$ is special in the sense that the spinorial representation has the same dimension as the vector representation. The symmetry behind this relation is called *triality symmetry*. Argue why you could guessed such a symmetry by inspecting the Dykin diagram of SO(8). (*1 Point*)

10.2 The Ramond-Neveu-Schwarz superstring

Superstring theory has different formulations. The three main formalisms are: The *Ramond-Neveu-Schwarz* (RNS), the *Green-Schwarz* (GS) and the *pure spinor formalism*. In the RNS formalism two-dimensional worldsheet supersymmetry is manifest. In the GS and in the pure spinor formalism it is the d-dimensional spacetime which is supersymmetric.
In the *Ramond-Neveu-Schwarz* formalism the supersymmetric partners of the bosonic fields $X^\mu(\tau,\sigma)$ described by the Polyakov action S_P are d free fermions (spinors) on the worldsheet with a target space vector index $\psi^\mu(\tau,\sigma)$. Their corresponding action is called the *Dirac action*. The sum of these actions in the *superconformal gauge* is given by

$$S_{\text{RNS}} = -\frac{1}{2\pi}\int d^2\sigma\left(\partial^\alpha X^\mu \partial_\alpha X_\mu + i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right)\ , \tag{1}$$

where ρ^α with $\alpha = 0,1$ form a representation of the $2d$ Clifford algebra, $\psi^\mu = \begin{pmatrix}\psi^\mu_+\\ \psi^\mu_-\end{pmatrix}$ is a $2d$ Majorana spinor (a $2d$ real spinor) in the vector representation of the Lorentz group $\mathrm{SO}(d-1,1)$ and $\bar{\psi} = \psi^\dagger\rho^0 = \psi^T\rho^0$ is the Dirac conjugate of ψ. Since a Majorana spinor in $2d$ is real we have $\left(\psi^\mu_+\right)^* = \psi^\mu_+$ and $\left(\psi^\mu_-\right)^* = \psi^\mu_-$.

a) Check that

$$\rho^0 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \quad \text{and} \quad \rho^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

satisfy the $2d$ Clifford algebra and evaluate the signature on the worldsheet. (*1 Point*)

b) Show that the action (1) in light-cone coordinates is given by

$$S_{\text{RNS}} = \frac{1}{\pi}\int d^2\sigma\left(2\partial_+ X\partial_- X + i\psi_-\partial_+\psi_- + i\psi_+\partial_-\psi_+\right)\ . \tag{2}$$

(*2 Points*)

c) Check that (1) is invariant (up to a total derivative) under the global worldsheet supersymmetry variations given by

$$\delta X^\mu = i\bar{\epsilon}\psi^\mu \quad \text{and} \quad \delta\psi^\mu = \rho^\alpha\partial_\alpha X^\mu\epsilon\ , \tag{3}$$

where $\epsilon = \begin{pmatrix}\epsilon_+\\ \epsilon_-\end{pmatrix}$ is an infinitesimal constant Majorana spinor.

Hint: Work out separately terms proportional to ϵ_- and ϵ_+. (*2 Points*)

d) Show that the supersymmetry variation (3) induces the conserved worldsheet supercurrents

$$j_+ = \psi^\mu_+\partial_+ X_\mu \quad \text{and} \quad j_- = \psi^\mu_-\partial_- X_\mu\ .$$

(*1 Point*)

e) Show that the non-zero components of the energy-momentum tensor are given by

$$T_{++} = \partial_+ X^\mu\partial_+ X_\mu + \frac{i}{2}\psi^\mu_+\partial_+\psi_{+\mu} \quad \text{and}$$
$$T_{--} = \partial_- X^\mu\partial_- X_\mu + \frac{i}{2}\psi^\mu\partial_-\psi_{-\mu}\ .$$

(*1 Point*)

f) Consider only the fermionic part of the action (2). Show that it is invariant under variation of $\psi_\pm$ with equations of motion $\partial_+\psi_- = 0$ and $\partial_-\psi_+ = 0$ from (2) describing left− and right movers which are Weyl conditions in $2d$. Therefore, $\psi_\pm$ are Majorana-Weyl spinors. Moreover, show that there is a boundary term left which is given by

$$\delta S_f = \int d\tau \left(\psi_+\delta\psi_+ - \psi_-\delta\psi_-\right)|_{\sigma=l} - \int d\tau \left(\psi_+\delta\psi_+ - \psi_-\delta\psi_-\right)|_{\sigma=0} \ .$$

(*2 Points*)

g) In the case of *open strings* show that the boundary term vanishes for $\psi_+^\mu = \pm\psi_-^\mu$ at each end of the string where the overall relative sign is a matter of convention. We choose without loss of generality $\psi_+^\mu|_{\sigma=0} = \psi_-^\mu|_{\sigma=0}$. Therefore, show that the vanishing of the boundary term translates to two distinct sectors: The Ramond boundary conditions $\psi_+^\mu|_{\sigma=l} = \psi_-^\mu|_{\sigma=l}$ which give rise to spacetime fermions and the Neveu-Schwarz boundary conditions $\psi_+^\mu|_{\sigma=0} = -\psi_-^\mu|_{\sigma=l}$ which give rise to spacetime bosons. (*1 Point*)

h) In the case of *closed strings* show that the boundary term vanishes for the periodicity conditions $\psi_\pm^\mu(\sigma) = \pm\psi_\pm^\mu(\sigma + l)$ where the positive sign (periodic boundary conditions) correspond to Ramond boundary conditions and the negative sign (anti-periodc boundary conditions) corresponds to Neveu-Schwarz boundary conditions. Furthermore, show that these give rise to four distinct closed string sectors: (NS-NS), (R-R) which give rise to spacetime bosons and (NS-R), (R-NS) which give rise to spacetime fermions. (*1 Point*)