Superstring Theory

Priv.-Doz. Dr. Stefan Förste und Christoph Nega

http://www.th.physik.uni-bonn.de/people/forste/exercises/strings19

Due date: 10.01.2020

–Homeworks–

11.1 Reduction to moduli of the string partition function

The string partition function for a worldsheet with the topology of a compact Riemann surface with genus g is given by

$$Z = \mathcal{N} \int \mathcal{D}h_{ab}\mathcal{D}X^{\mu}\exp\left[-\int \mathrm{d}^2\sigma\sqrt{h}\left(\frac{1}{2}h^{ab}\partial_a X^{\mu}\partial_b X_{\mu} + \frac{1}{4\pi}\lambda\mathcal{R}\right)\right] , \tag{1}$$

with $\mathcal{N}$ some normalization, (σ^1, σ^2) are coordinates on the worldsheet, $X^{\mu}(\sigma^1, \sigma^2)$ the mapping from the worldsheet to the target space, λ the string coupling, $\mathcal{R}$ the Ricci scalar of the worldsheet, $h = \det h_{ab}$ and h_{ab} a metric on the wordsheet.

a) Show that the partition function in (1) is equivalent to *(0,5 Points)*

$$Z_g = \mathcal{N}\mathrm{e}^{\lambda(2g-2)} \int \mathcal{D}h_{ab}\mathcal{D}X^{\mu}\exp\left(-\int \mathrm{d}^2\sigma\sqrt{h}\ \frac{1}{2}h^{ab}\partial_a X^{\mu}\partial_b X_{\mu}\right) . \tag{2}$$

The integral in (2) is highly divergent because one integrates infinitely many times over conformally equivalent surfaces. We need to extract the divergent part and only integrate over physically inequivalent metrics. Let $\mathcal{G}_g$ denote the space of admissible metrics on a compact Riemann surface with genus g. Then

$$\langle \delta h^{(1)}_{ab}, \delta h^{(2)}_{cd}\rangle = \int \mathrm{d}^2\sigma\sqrt{h}h^{ac}h^{bd}\delta h^{(1)}_{ab}\delta h^{(2)}_{cd}$$

defines a scalar $\langle\cdot,\cdot\rangle$ product of two infinitesimal variations $\delta h^{(1)}_{ab}$ and $\delta h^{(2)}_{ab}$ of the metric h_{ab}. More precise, $\delta h^{(1)}_{ab}$ and $\delta h^{(2)}_{ab}$ are elements of the tangent space $T_h(\mathcal{G}_g)$ at the the point $h_{ab} \in \mathcal{G}_g$.

b) Recall that Weyl transformations and diffeomorphisms of the metric do not change the physical results and are therefore symmetries of the worldsheet. Show that under the combined action of

- a Weyl scaling $h_{ab} \mapsto \mathrm{e}^{\phi}h_{ab}$
- and a diffeomorphism generated by a vector field $\vec{v}$

the infinitesimal variation of h_{ab} is given by

$$\delta h_{ab} = \delta\phi h_{ab} + \nabla_a v_b + \nabla_b v_a \ .$$

Split δh_{ab} into a trace part $\delta h_{ab}^{\mathrm{W}}$ and a symmetric traceless part $\delta h_{ab}^{\mathrm{D}}$ and show that the the norm of δh_{ab} is given by

$$||\delta h||^2 = ||\delta h^{\mathrm{W}}||^2 + ||\delta h^{\mathrm{D}}||^2 \ .$$

Write down the explicit form of $\delta h_{ab}^{\mathrm{W}}$ and $\delta h_{ab}^{\mathrm{D}}$.

Hint: Define an operator P which maps vectors $\vec{v}$ to second-rank symmetric traceless tensors $v_a \to (P\vec{v})_{ab}$. *(3 Points)*

c) Conformal killing vectors (CKV) $\vec{v}_{\mathrm{CKV}}$ fulfill the condition

$$\nabla_a v_b + \nabla_b v_a = \lambda h_{ab} \quad \text{with} \quad \lambda \in \mathbb{R} \ .$$

Show that CKV are elements of $\ker(P)$ and can be replaced by Weyl scalings. *(1 Point)*

We are interested in the subspace $\mathcal{M}_g \subset \mathcal{G}_g$, containing all conformal equivalence classes of metrics, called the *moduli space*. Denoting the set of Weyl scalings as $Weyl$ and the set of diffeomorphisms as $Diff$, the moduli space is represented as[1]

$$\mathcal{M}_h \sim \frac{\mathcal{G}_g}{Weyl \times Diff} \ .$$

In general an infinitesimal variation of a metric $h(t_i) \in \mathcal{G}_g$ is given by

$$\delta h_{ab} = \delta h_{ab}^{\mathrm{W}} + \delta h_{ab}^{\mathrm{D}} + \delta t_i \frac{\partial}{\partial t_i} h_{ab} \ ,$$

where t_i are the moduli parameter.

d) Split the term $\delta t_i \frac{\partial}{\partial t_i} h_{ab}$ into a trace and a traceless symmetric part and define an operator T_{ab}^i which acts on δt_i and maps it to the traceless symmetric term of $\delta t_i \frac{\partial}{\partial t_i} h_{ab}$. *(2 Points)*

In order to integrate in (2) only over physically inequivalent metrics we need to find an appropriate gauge slice in $\mathcal{G}_g$. Therefore, we first need to find a slice $\tilde{\mathcal{G}}_g$ which contains all equivalence classes of metrics related by Weyl transformations. Then the wanted gauge slice lies in $\tilde{\mathcal{G}}_g$ and is chosen in such a way that a transformation $\exp(P\vec{v})$ on a point $\tilde{g}_{ab} \in \{\text{gauge slice}\}$ leads to a point $\hat{h}_{ab}$ still in $\tilde{\mathcal{G}}_g$ but no longer in the wanted gauge slice.

e) Consider a point $\tilde{h}_{ab} \in \tilde{\mathcal{G}}_g$. Argue whether it is possible to act on $\tilde{h}_{ab}$ with diffeomorphism in such a way, that it leaves the slice $\tilde{\mathcal{G}}_g$. *(1 Point)*

f) Let us denote an infinitesimal variation changing the conformal equivalence class by $\delta h_{ab}^{\perp}$. It is therefore a tangent vector in the tangent space of $\mathcal{M}_g$. Argue why $\delta h_{ab}^{\perp}$ must be traceless. Show that $\delta h_{ab}^{\perp} \in \ker(P^\dagger)$.

Hint: Compute the angle between the tangent vector $(P\vec{v})_{ab}$ and $\delta h_{ab}^{\perp}$. *(1,5 Points)*

[1] Actually, it is the semidirect product of $Weyl$ and $Diff$.

g) Let ψ^α_{ab}, $\alpha = 1, ..., \dim(\ker(P^\dagger))$ be an orthonormal basis for $\ker(P^\dagger)$. Decompose $T^i_{ab}\delta t_i$ into a linear combination of basis vectors of $\ker(P^\dagger)$ and vectors of range(P). This yields

$$T^i_{ab}\delta t_i = \langle\psi^\alpha, T^i\rangle\psi^\alpha_{ab}\delta t_i + \frac{\langle P\vec{v}, T^i\rangle}{||P\vec{v}||^2}(P\vec{v})_{ab}\delta t_i \ .$$

Show that the norm of δh_{ab} is given by

$$||\delta h||^2 = ||\delta\tilde{\phi}||^2 + ||P\vec{\tilde{v}}||^2 + \langle\psi^\alpha, T^i\rangle\langle\psi^\alpha, T^j\rangle\delta t_i\delta t_j \ ,$$

with

$$\delta\tilde{\phi} = \delta\phi + \nabla_c v^c + \frac{1}{2}\left(h^{cd}\delta t_i\frac{\partial}{\partial t_i}h_{cd}\right) \quad \text{and} \quad \vec{\tilde{v}} = \left(1 + \frac{\langle P\vec{v}, T^i\delta t_i\rangle}{||P\vec{v}||^2}\right)\vec{v} \ .$$

(2 Points)

In order to change the path integral variables from h_{ab} to ϕ, $\vec{v}$ and t_i we use the relation

$$\begin{aligned} 1 &= \int \mathcal{D}h_{ab}\, \exp(-||\delta h||^2/2) \\ &= J\int \mathcal{D}\tilde{\phi}\mathcal{D}v'^a \mathrm{d}t^1\cdots\mathrm{d}t^n \exp\left(-\left[||\delta\tilde{\phi}||^2 + ||P\tilde{v}'||^2 + \langle\psi^\alpha, T^i\rangle\langle\psi^\alpha, T^j\rangle\delta t_i\delta t_j\right]/2\right) \end{aligned} \tag{3}$$

to calculate the Jacobian J. Notice that $\vec{v}'$ denotes elements from range(P). Since elements from $\ker(P)$ are orthogonal to $\vec{v}'$ we can decompose the volume of the diffeomorphism group V_{Diff} into $V^\perp_{Diff} \times V^{\mathrm{CKV}}_{Diff}$. Let χ^i, $i = 1, ..., \dim(\ker(P))$ be a basis for $\ker(P)$, then one can show that

$$V^\perp_{Diff} = V_{Diff}\left(\det\langle\chi^i, \chi^j\rangle\right)^{-1/2} \ .$$

From (3) one can show that the Jacobian for the path integral should be given by $J = \det^{1/2}(P^\dagger P)\frac{\det\langle\psi^i, T^j\rangle}{(\det\langle\psi^i,\psi^j\rangle)^{\frac{1}{2}}}$.

h) Show that

$$\int \mathcal{D}h_{ab} \to V_{Diff}\int \mathcal{D}\phi \mathrm{d}t^1\cdots\mathrm{d}t^n\left(\frac{\det(P^\dagger P)}{\det\langle\chi^i, \chi^j\rangle}\right)^{1/2}\frac{\det\langle\psi^i, T^j\rangle}{(\det\langle\psi^i, \psi^j\rangle)^{\frac{1}{2}}} \ .$$

(0,5 Points)

i) Express the number of real moduli n by the genus g for a compact Riemann surface with no crosscaps and $g \geq 2$.

<u>*Hint:*</u> There are no CKV for compact Riemann surfaces with $g \geq 2$. *(1 Point)*

In the critical dimension ($D = 26$ for the bosonic string) the integrand becomes independent from ϕ and the integral $\int \mathcal{D}\phi = V_{Conf}$ can be absorbed into the normalization. It can be shown that the integral over the mappings X^μ is given by

$$\int \mathcal{D}X^\mu \exp\left(-\int \mathrm{d}^2\sigma\sqrt{h}\,\frac{1}{2}h^{ab}\partial_a X^\mu\partial_b X_\mu\right) = \mathcal{V}\left(\frac{\int \mathrm{d}^2\sigma\sqrt{h}}{2\pi}\right)^{13}(\det\Delta_h)^{-13} \ ,$$

with $\mathcal{V}$ the volume of space time and $\Delta_h = -\frac{1}{\sqrt{h}}\partial_a\sqrt{h}h^{ab}\partial_b$. Putting the previous results together we find that the partition function in (1) can be expressed by

$$Z_g = \mathcal{V}\mathrm{e}^{\lambda(2g-2)}\int_{\mathcal{M}_g}\mathrm{d}t^1...\mathrm{d}t^n\left(\frac{\det(P^\dagger P)}{\det\langle\chi^i, \chi^j\rangle}\right)^{1/2}\frac{\det\langle\psi^i, T^j\rangle}{\det^{1/2}\langle\psi^i, \psi^j\rangle}\left(\frac{2\pi}{\int \mathrm{d}^2\sigma\sqrt{h}}\det\Delta_h\right)^{-13} \ .$$

j) Now we have the general expression for the partition function of a compact Riemann surface. We want to analyze more carefully the torus partition function Z_1 meaning that the worldsheet has the topology of a torus.

(i) Show that $\chi^1 = (1,0)^{\mathrm{T}}$ and $\chi^2 = (0,1)^{\mathrm{T}}$ are a possible choice for $\ker(P)$. *(0,5 Points)*

(ii) Show that T^i_{ab} are given by

$$T^1_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 2\tau_1 \end{pmatrix} \quad \text{and} \quad T^2_{ab} = \frac{1}{\tau_2}\begin{pmatrix} -1 & -\tau_1 \\ -\tau_1 & \tau_2^2 - \tau_1^2 \end{pmatrix} ,$$

where τ_1 and τ_2 are the moduli of the torus with the $h_{ab} = |\mathrm{d}\sigma^1 + (\tau_1 + i\tau_2)\mathrm{d}\sigma^2|^2$. Explain why T^1_{ab}, T^2_{ab} form a possible basis for $\ker(P^\dagger)$.

<u>*Hint*:</u> Since the metric is flat we have $(P^\dagger T^i)_b = -2\partial^a T^i_{ab}$. *(1,5 Points)*

(iii) Next calculate $\det\langle\chi^i, \chi^j\rangle$ and $\frac{\det\langle\psi^i, T^j\rangle}{\det^{1/2}\langle\psi^i,\psi^j\rangle}$ and show that $\det(P^\dagger P) = (\det(2\Delta_h))^2$.

<u>*Hint*:</u> First show $(P^\dagger P)_{ab} v^b = 2\delta_{ab}\Delta_h v^b$. *(1 Point)*

(iv) Use $\det(2\Delta_h) = \frac{1}{2}\det(2)\det(\Delta_h)$ and compute Z_1. You should arrive at

$$Z_1 = \mathcal{V}\int_{\mathcal{M}_1} \mathrm{d}^2\tau \frac{\tau_2^{10}}{(2\pi)^{13}} \left(\det(\Delta_h)\right)^{-12} \det(2) \ .$$

(1 Point)

The term $\det(2)$ can be absorbed into a counterterm by modifying the action. The computation of $\det(\Delta_h)$ would lead to

$$\det \Delta_h = \tau_2^2 \mathrm{e}^{-\pi\tau_2/3} \left| \prod_{n=1}^{\infty} 1 - \mathrm{e}^{2i\pi n\tau} \right|^4 .$$

Plugging it into Z_1 we arrive at the final result for the torus partition function

$$\begin{aligned} Z_1 &= \int_{\mathcal{M}_1} \frac{\mathrm{d}^2\tau}{2\pi\tau_2^2} (2\pi\tau_2)^{-12} \mathrm{e}^{4\pi\tau_2} \left| \prod_{n=1}^{\infty} 1 - \mathrm{e}^{2i\pi n\tau} \right|^{-48} \\ &= \frac{1}{2}\int_{\mathcal{F}_{\mathrm{PSL}(2,\mathbb{Z})}} \frac{\mathrm{d}^2\tau}{2\pi\tau_2^2} (2\pi\tau_2)^{-12} \mathrm{e}^{4\pi\tau_2} \left| \prod_{n=1}^{\infty} 1 - \mathrm{e}^{2i\pi n\tau} \right|^{-48} , \end{aligned}$$

where $\mathcal{F}_{\mathrm{PSL}(2,\mathbb{Z})}$ is the fundamental domain of the modular group $\mathrm{PSL}(2,\mathbb{Z})$. Notice that integrating over $\mathcal{F}_{\mathrm{PSL}(2,\mathbb{Z})}$ leaves an unfixed residual gauge freedom given by the diffeomorphism $\sigma^1 \to -\sigma^1$, $\sigma^2 \to -\sigma^2$. Therefore a factor of $\frac{1}{2}$ is necessary to remove the overcounting.

From exercise 9.2 it follows that the torus partition function is invariant under modular transformations.

11.2 Physical interpretation of the torus partition function

The effective action for a massive free scalar field ϕ is defined by

$$\mathrm{e}^{-\Gamma} = \int \mathcal{D}\phi \mathrm{e}^{-S[\phi]} \quad \text{with} \quad S[\phi] = \frac{1}{2}\int \mathrm{d}^d x \phi(-\Box + m^2)\phi \ .$$

The exponent Γ is proportional to the vacuum energy E_{vac}

$$\Gamma = \mathcal{V} E_{\text{vac}} \ ,$$

where $\mathcal{V}$ is the volume of spacetime.

a) Compute $\mathrm{e}^{-\Gamma}$.
   <u>*Hint:*</u> The result is $\mathrm{e}^{\Gamma} = \left(\det(-\Box + m^2)\right)^{-1/2}$. *(2 Points)*

b) Show that

$$E_{\text{vac}} = -\frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \log(k^2 + m^2) \ ,$$

   where k^μ is the momentum in the μ-th direction.
   <u>*Hint:*</u> Use $\mathbb{1} = \int \mathrm{d}^d k |k\rangle\langle k|$. *(1 Point)*

c) In string theory a string has many excitations generating many massive fields. Therefore, the vacuum energy for a bosonic string must be the sum over all excitations

$$E_{\text{vac}} = -\frac{1}{2}\sum_i \int \frac{\mathrm{d}^d k}{(2\pi)^d} \log(k^2 + m_i^2) \ .$$

   Use the properties $\log A = -\lim_{\epsilon\to 0} \frac{\mathrm{d}}{\mathrm{d}\epsilon} A^{-\epsilon} = -\lim_{\epsilon\to 0} \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[\epsilon \int \frac{\mathrm{d}t}{t^{1-\epsilon}} \mathrm{e}^{-2\pi t A}\right]$ to show that the vacuum energy can be writen as a trace over the Hilbert space $\mathcal{H}$

$$E_{\text{vac}} \propto \int \frac{\mathrm{d}t}{t} \operatorname{Tr}_{\mathcal{H}}\left[\, \mathrm{e}^{\,\cdots} \right] \ .$$

(0,5 Points)