Superstring Theory

Priv.-Doz. Dr. Stefan Förste und Christoph Nega

http://www.th.physik.uni-bonn.de/people/forste/exercises/strings19

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–Homeworks–

13.1 Spin structures, GSO projection and type IIA/IIB spectrum

In this exercise we investigate closed superstring theories with $d = 10$ of type IIA and IIB be analyzing their fermionic partition function. Consider all string states $|state\rangle$ living in the Hilbert space $\mathcal{H}$ of the theory. The partition function is given by the following trace over the string states

$$Z(\tau) = \sum_{|state\rangle \in \mathcal{H}} \langle state| \mathrm{e}^{-\tau_2 l H} \mathrm{e}^{i\tau_1 l P} |state\rangle \ ,$$

where H is the Hamiltonian, P is the momentum operator and $\tau = \tau_1 + i\tau_2$ is the complex structure of the torus. In the following, we consider only the fermionic partition function. The Hamiltonian for the closed string is a sum over the Hamiltonian for the left- and right-moving string states

$$H = H_L + H_R \ ,$$

where the Hamiltonian of the left-movers in light-cone gauge is given by their NS and R sectors

$$H_{\mathrm{NS},L} = \sum_{i=2}^{9} \sum_{r=0}^{\infty} \left(r + \frac{1}{2} \right) b^i_{-r-\frac{1}{2}} b^i_{r+\frac{1}{2}} - \frac{1}{6} \ ,$$

$$H_{\mathrm{R},L} = \sum_{i=2}^{9} \sum_{r=0}^{\infty} r b^i_{-r} b^i_r + \frac{1}{3} \ .$$

There are similar expressions for the Hamiltonian of the right-movers with oscillators carrying a $\tilde{}$. The momentum operator is $P = P_L - P_R$, where $N_{\mathrm{NS}} = \sum_{r=0}^{\infty} \left(r + \frac{1}{2} \right) b^i_{-r-\frac{1}{2}} b^i_{r+\frac{1}{2}}$ is the number operator in the Neveu-Schwarz sector and $N_{\mathrm{R}} = \sum_{r=0}^{\infty} r b^i_{-r} b^i_r$ is the number operator in the Ramond sector.

a) Show that the closed string partition function $Z(\tau)$ splits into a product of a left- and a right-handed partition function

$$Z(\tau) = \mathrm{Tr}_{\mathcal{H}_L} \left[q^{N+E_0} \right] \cdot \mathrm{Tr}_{\mathcal{H}_R} \left[\bar{q}^{\tilde{N}+\tilde{E}_0} \right] \ , \tag{1}$$

where the trace is taken over the Hilbert space of left- and right-moving string states, E_0 and $\tilde{E}_0$ are their respective zero-point energies, $l = 2\pi$ and $q = \mathrm{e}^{2\pi i \tau}$. *(2 Points)*

b) Consider know only the left-moving states. Show that the partition functions in the Neveu-Schwarz and in the Ramond setor are given respectively by

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{NS}}}\left[q^{N_{\mathrm{NS}}+E_{0,\mathrm{NS}}}\right] = \frac{\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4}{\eta^4} \quad \text{and} \quad \mathrm{Tr}_{\mathcal{H}_{\mathrm{R}}}\left[q^{N_{\mathrm{R}}+E_{0,\mathrm{R}}}\right] = \frac{\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^4}{\eta^4} ,$$

where η is the Dedekind function and $\vartheta[\ldots]$ are the Jacobi-theta functions defined in the lecture. They can be also find in Chapter 9 in the book by Blumenhagen, Lüst and Theisen. Similar expressions hold for the right-moving states with tilded modes.

(4 Points)

c) Construct the partition function with all possible string states by glueing left- and right-moving states together making use of (1). Moreover, check whether the partition function is modular invariant.

You should then conclude that something must be wrong in the way we constructed the closed string partition function. To cure the problem, we must investigate in more detail the boundary conditions for fermions on the worldsheet.

d) Consider a string worldsheet with the topology of a sphere on which bosonic and fermionic states live. The rotation group on the manifold is SO(2), which is isomorphic to U(1). The representations of SO(2) are given by specifying the eigenvalues of the unique generator W of SO(2). Show that a vector field V^μ decomposes into components V^+ and V^- of $W = 1$ and $W = -1$, respectively. A two-component spinor field b^A has components b^+ and b^- with $W = -1/2$ and $W = 1/2$, respectively. If it is Majorana state how many degrees of freedom it has. *(1.5 Points)*

e) Consider V^+ parallel transported around a closed path γ on the sphere. V^+ then picks up a phase $e^{i\alpha}$. Now explain what happens when b^+ is parallel transported around γ. Explain further why one could naively assume that there would be a sign ambiguity for the parallel transported b^+. *(1.5 Points)*

f) Now shrink the path γ to a point and show that the sign must be $+1$. Describe what happens if γ encloses a handle on the wolrdsheet[1]. The different signs are called *spin structures*.

(1 Point)

g) Determine all spin structures on a two-dimensional torus. Parameterize the torus by the worldsheet coordinates τ and σ and identify the boundary conditions for the Neveu-Schwarz and the Ramond sectors. *(2 Points)*

To incorporate the spin structure in the τ direction, one introduces the *GSO projection operator* $(-1)^F$, where F is the fermion number operator. In the Neveau-Schwarz sector it is given by

$$F_{\mathrm{NS}} = \sum_{r=0}^{\infty} b^i_{-r-\frac{1}{2}} b^i_{r+\frac{1}{2}} .$$

In the Ramond sector the GSO operator takes the form

$$(-1)^{F_{\mathrm{R}}} = \gamma^{11}(-1)^{\sum_{r=1}^{\infty} b^i_{-r} b^i_r} ,$$

[1]Strictly speaking this is not possible for a sphere, but for higher genus worldsheets only. Here we want only a qualitative analysis and you should neglect this issue.

where $\gamma^{11} = \gamma^0\gamma^1 \cdots \gamma^9$ is the ten-dimensional chirality operator. The GSO operator anticommutes with worldsheet fermions

$$(-1)^F b^i_\mu = -b^i_\mu (-1)^F \ .$$

h) Calculate the partition functions for the left-moving and right-moving Neveu-Schwarz and Ramond sectors with the insertion of the GSO projection $\frac{1\pm(-1)^F}{2}$. For the Neveu-Schwarz sector the GSO projection takes in the left- and right-moving sectors the following forms

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{NS},L}}\left[q^{N_{\mathrm{NS}}+E_{0,\mathrm{NS}}}\frac{1-(-1)^{F_{\mathrm{NS}}}}{2}\right] \quad \text{and} \quad \mathrm{Tr}_{\mathcal{H}_{\mathrm{NS},R}}\left[q^{\tilde{N}_{\mathrm{NS}}+\tilde{E}_{0,\mathrm{NS}}}\frac{1-(-1)^{F_{\mathrm{NS}}}}{2}\right] \ .$$

In the left-moving Ramond sector the GSO projection takes the same form as in the Neveu-Schwarz sector

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{R},L}}\left[q^{N_{\mathrm{R}}+E_{0,\mathrm{R}}}\frac{1-(-1)^{F_{\mathrm{R}}}}{2}\right] \ ,$$

but in the right-moving Ramond sector the GSO projection can take two different signs

$$\begin{aligned} \mathrm{A}: \quad & \mathrm{Tr}_{\mathcal{H}_{\mathrm{R},R}}\left[q^{\tilde{N}_{\mathrm{R}}+\tilde{E}_{0,\mathrm{R}}}\frac{1+(-1)^{F_{\mathrm{R}}}}{2}\right] \\ \mathrm{B}: \quad & \mathrm{Tr}_{\mathcal{H}_{\mathrm{R},R}}\left[q^{\tilde{N}_{\mathrm{R}}+\tilde{E}_{0,\mathrm{R}}}\frac{1-(-1)^{F_{\mathrm{R}}}}{2}\right] \ . \end{aligned}$$

Identify the spin structures for the different pieces of the partition function and show that the total fermionic partition function vanishes. *(5 Points)*

The possibility to choose the GSO projection in the right-moving Ramond sector leads to two different closed superstring theories called *type IIA* and *type IIB*.