Superstring Theory

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http://www.th.physik.uni-bonn.de/people/forste/exercises/strings19

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–HOMEWORKS–

2.1 Worldsheet light-cone coordinates

In exercise 1.3 we have seen that worldsheet reparametrization invariance of the Polyakov action S_P can be used to locally fix the coordinates to the so-called *conformal gauge* with $h_{\alpha\beta} = \Omega^2(\sigma,\tau)\eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is the two-dimensional Minkowski metric, i.e. $ds^2 = -d\tau^2 + d\sigma^2$.
Let us define the so-called *worldsheet light-cone coordinates* by

$$\sigma^{\pm} = \tau \pm \sigma \ ,$$

such that the worldsheet metric is given by

$$ds^2 = -\Omega^2(\sigma^+, \sigma^-) d\sigma^+ d\sigma^- \ .$$

Weyl invariance of the Polyakov action S_P can be further used to locally set $h_{\alpha\beta} = \eta_{\alpha\beta}$.

a) Express the derivatives $\partial_+ := \frac{\partial}{\partial\sigma^+}$ and $\partial_- := \frac{\partial}{\partial\sigma^-}$ through ∂_τ and ∂_σ. (*1 Point*)

b) Find the components of the Minkowski metric $\eta_{\alpha\beta}$ in light-cone coordinates. (*1 Point*)

The energy-momentum tensor of the worldsheet theory is defined as

$$T_{\alpha\beta} = \frac{4\pi}{\sqrt{-h}} \frac{\delta S_P}{\delta h^{\alpha\beta}} \ .$$

c) Compute the energy-momentum tensor $T_{\alpha\beta}$. Argue that $T_{\alpha\beta}$ has to vanish. (*2 Points*)

d) Show in two different ways that the energy-momentum tensor of the worldsheet theory is traceless. (*2 Points*)

e) Compute the components of the energy-momentum tensor in light-cone coordinates. (*2 Points*)

2.2 Classical string equations of motion and boundary conditions

In this exercise we want to analyze the classical string equations of motion in view of different boundary conditions one can impose.
Consider the range of the worldsheet coordinates

$$\tau \in (\tau_i, \tau_f) \quad \text{and} \quad \sigma \in [0, l) \ .$$

a) Show that the Polyakov action in conformal gauge is given by

$$S_P = \frac{T}{2} \int D^2\sigma \left(\dot{X}^2 - X'^2 \right) \ , \tag{1}$$

where $\dot{}$ and $'$ denote derivatives with respect to τ and σ, respectively. (*1 Point*)

b) Vary the action S_P from (1) with respect to X^μ under the assumption $\delta X^\mu(\tau_i) = \delta X^\mu(\tau_f) = 0$ in order to show that the equations of motion are given by

$$\left(\partial_\sigma^2 - \partial_\tau^2 \right) X^\mu = 0 \ .$$

Additionally, show that there is still a boundary term of the form

$$-T \int d\tau \left(X'_\mu \delta X^\mu \big|_{\sigma=l} - X'_\mu \delta X^\mu \big|_{\sigma=0} \right) \tag{2}$$

left. (*2 Points*)

c) Show that there are three different possibilities such that the boundary term (2) vanishes. These are

i) $X^\mu(\sigma, \tau) = X^\mu(\sigma + l, \tau)$,

ii) $X'_\mu(\sigma, \tau) = 0 \quad \text{for } \sigma = 0, l$,

iii) $X^\mu\big|_{\sigma=0} = X_0^\mu = \text{const.} \quad \text{and} \quad X^\mu\big|_{\sigma=l} = X_l^\mu = \text{const.}$.

Moreover, comment on the physical interpretation of each of the three boundary conditions. (*3 Points*)

2.3 Global Poincaré invariance and Poincaré algebra

In exercise 1.3 we showed that the Polyakov action is invariant under global Poincaré transformations. Furthermore, we computed using the Noether theorem the associated conserved currents, namely the energy-momentum current P_μ^α and the angular momentum current $J_{\mu\nu}^\alpha$.

a) Show that

$$J_{\mu\nu}^\alpha = X_\mu P_\nu^\alpha - X_\nu P_\mu^\alpha \ .$$

(*1 Point*)

Integrating the currents P_μ^α and $J_{\mu\nu}^\alpha$ over a space-like section (i.e. for fixed τ) of the world-sheet we obtain the total conserved charges which are momentum and angular momentum.
In the following, we work again in conformal gauge.

b) Show that indeed the total momentum $P_\mu = \int_0^l d\sigma P^\tau_\mu$ and the total angular momentum $J_{\mu\nu} = \int_0^l d\sigma J^\tau_{\mu\nu}$ are conserved for the closed string. Decide for which boundary conditions of the open string total momentum and total angular momentum are conserved. Comment on the physical interpretation. *(2 Points)*

c) Show that P^μ and $J^{\mu\nu}$ generate the Poincaré algebra

$$
\begin{aligned}
\{P^\mu, P^\nu\}_{\text{P.B.}} &= 0 \ , \\
\{P^\mu, J^{\rho\sigma}\}_{\text{P.B.}} &= \eta^{\mu\sigma}\ P^\rho - \eta^{\mu\rho}\ P^\sigma \ , \\
\{J^{\mu\nu}, J^{\rho\sigma}\}_{\text{P.B.}} &= \eta^{\mu\rho}\ J^{\nu\sigma} + \eta^{\nu\sigma}\ J^{\mu\rho} - \eta^{\nu\rho}\ J^{\mu\sigma} - \eta^{\mu\sigma}\ J^{\nu\rho} \ .
\end{aligned}
$$

Hint: Use the equal time Poisson brackets

$$
\begin{aligned}
\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\}_{\text{P.B.}} &= \{\dot{X}^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\}_{\text{P.B.}} = 0 \ , \\
\{X^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\}_{\text{P.B.}} &= \frac{1}{T}\ \eta^{\mu\nu}\ \delta(\sigma - \sigma') \ .
\end{aligned}
$$

(3 Points)