Physikalisches Institut
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Theoretische Physik

Exercise Sheet 3
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Superstring Theory

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http://www.th.physik.uni-bonn.de/people/forste/exercises/strings19

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–Homeworks–

3.1 Oscillator expansions for the classical string

In exercise 2.2 we have derived the equations of motion for the classical string in conformal gauge, i.e. $\left(\partial_\sigma^2 - \partial_\tau^2\right) X^\mu$. The vanishing of the surface term followed after imposing any of the three following boundary conditions

i) Closed String: $X^\mu(\sigma,\tau) = X^\mu(\sigma + l, \tau)$,

ii) Open String (Neumann): $X'_\mu(\sigma,\tau) = 0 \quad \text{for } \sigma = 0, l$,

iii) Open String (Dirichlet): $X^\mu|_{\sigma=0} = X_0^\mu = \text{const.}$ and $X^\mu|_{\sigma=l} = X_l^\mu = \text{const.}$.

We make a product ansatz for a solution of the equations of motion to any of the d spacetime coordinates X^μ with $\mu = 0, \ldots, d-1$ given by

$$X^\mu(\sigma,\tau) = f(\sigma) g(\tau) \ .$$

We start our discussion with the *closed string*.

a) Show that the functions $f(\sigma)$ and $g(\tau)$ have to satisfy

$$\frac{\partial^2 f(\sigma)}{\partial \sigma^2} = k f(\sigma) \, , \quad \frac{\partial^2 g(\tau)}{\partial \tau^2} = k g(\tau), \quad k = -\frac{4m^2\pi^2}{l^2}, \quad m \in \mathbb{Z} \ .$$

Solve these differential equations for $m = 0$ as well as for $m \in \mathbb{Z}\backslash\{0\}$. Out of this write the full solution $X^\mu(\sigma,\tau)$. (*2 Points*)

b) Argue that the general solution for a *closed string* splits into left-movers and right-movers

$$X^\mu(\sigma,\tau) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-) \ .$$

With a convenient normalization (inspired from the analysis in previous part) the left-movers and the right-movers are respectively given by

$$X_L^\mu(\sigma^+) = \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{l} p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu \mathrm{e}^{-\frac{2\pi}{l} i n \sigma^+} \ ,$$

$$X_R^\mu(\sigma^-) = \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{l} p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu \mathrm{e}^{-\frac{2\pi}{l} i n \sigma^-} \ .$$

In these equations $n \in \mathbb{Z}\backslash\{0\}$, α_n^μ and $\tilde{\alpha}_n^\mu$ are arbitrary Fourier modes with $\alpha_n^\mu, \tilde{\alpha}_n^\mu$ positive-frequency modes for $n < 0$ and $\alpha_n^\mu, \tilde{\alpha}_n^\mu$ negative-frequency modes for $n > 0$ and c^μ is an arbitrary parameter which can be set to zero if the zero-mode part of the expansions is left-right symmetric. *(1 Point)*

c) Take the solution from part a) and give conditions which have to be imposed on $x^\mu, p^\mu, \alpha_n^\mu$ and $\tilde{\alpha}_n^\mu$ in order to make X^μ real. *(1 Point)*

d) We define $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$. Compute the following quantities in terms of the oscillator modes appearing in the left-movers $X_L^\mu(\sigma^+)$ and the right-movers $X_R^\mu(\sigma^-)$

   i) total spacetime momentum: $P^\mu = \frac{1}{2\pi\alpha'} \int_0^l d\sigma \dot{X}^\mu$,

   ii) center of mass position: $q^\mu = \frac{1}{l} \int_0^l d\sigma X^\mu$,

   iii) total angular momentum: $J^{\mu\nu} = \frac{1}{2\pi\alpha'} \int_0^l d\sigma \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu \right)$.

   Furthermore, explain the physical interpretation of p^μ and x^μ for the string. *(3 Points)*

Now we want to obtain the oscillator expansions for *open strings*.

e) Argue that open string solutions have only one set of oscillator modes. *(1 Point)*

f) Show that the mode expansion for *open strings* with *Neumann* boundary conditions at both ends of the string, i.e. for $\sigma = 0$ and $\sigma = l$, is given by

$$X^\mu(\sigma, \tau) = x^\mu + \frac{2\pi\alpha'}{l} p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-i\frac{\pi}{l}n\tau} \cos\left(\frac{n\pi\sigma}{l}\right) .$$

(1 Point)

g) Show that the mode expansion for *open strings* with *Dirichlet* boundary conditions at both ends of the string, i.e. for $\sigma = 0$ and $\sigma = l$, is given by

$$X^\mu(\sigma, \tau) = x_0^\mu + \frac{1}{l}\left(x_l^\mu - x_0^\mu\right)\sigma + \sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-i\frac{\pi}{l}n\tau} \sin\left(\frac{n\pi\sigma}{l}\right) .$$

(1 Point)

3.2 Poisson brackets for the classical closed string

In this exercise we want to find the Poisson brackets for the oscillator modes $\alpha_n^\mu, \tilde{\alpha}_n^\mu$. Recall the equal time Poisson brackets

$$\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\}_{\text{P.B.}} = \{\dot{X}^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\}_{\text{P.B.}} = 0 ,$$
$$\{X^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\}_{\text{P.B.}} = \frac{1}{T} \eta^{\mu\nu} \delta(\sigma - \sigma') .$$

Use the oscillator expansion for the closed string from the previous exercise 3.1 to show that the Poisson brackets for the modes are given by

$$\{\alpha_n^\mu, \alpha_m^\nu\}_{\text{P.B.}} = \{\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu\}_{\text{P.B.}} = -in\eta^{\mu\nu}\delta_{m+n,0} , \quad \{\alpha_n^\mu, \tilde{\alpha}_m^\nu\}_{\text{P.B.}} = 0$$
$$\{x^\mu, x^\nu\}_{\text{P.B.}} = \{p^\mu, p^\nu\}_{\text{P.B.}} = 0 , \qquad \{x^\mu, p^\nu\}_{\text{P.B.}} = \eta^{\mu\nu} .$$

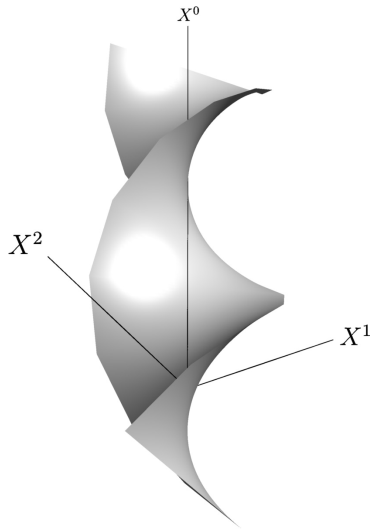
<u>*Hint*</u>: Express α_n^μ and $\tilde{\alpha}_n^\mu$ as linear combinations of $X^\mu(\sigma,\tau)$ and $\dot{X}^\mu(\sigma,\tau)$ for fixed τ. Then you can use the equal time Poisson brackets. *(4 Points)*

3.3 A spaghetti stick as solution to the string equations of motion

We consider a with constant angular velocity rotating spaghetti stick in the X^1-X^2-plane parametrized by

$$X^0 = A\tau \ , \quad X^1 = A\cos\tau\cos\sigma \ , \quad X^2 = A\sin\tau\cos\sigma \ , \quad X^i = 0 \quad \text{for } i = 3, \ldots, d-1 \ .$$

The worldsheet of this configuration looks like



Throughout this exercise we want to show that this configuration is an open string solution and compute its characteristics.

a) Verify that the rotating spaghetti stick is indeed an open string solution. (*1 Point*)

b) Determine the boundary conditions of this configuration and the speed of the string's endpoints. (*1 Point*)

c) Compute the energy $M := P^0$ of the spaghetti stick. (*1 Point*)

d) Compute the angular momentum $J := |J^{12}|$. (*1 Point*)

e) Compute the quantity $\frac{J}{M^2}$. (*1 Point*)