Physikalisches Institut
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Theoretische Physik

Exercise Sheet 4
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Superstring Theory

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`http://www.th.physik.uni-bonn.de/people/forste/exercises/strings19`

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–Homeworks–

4.1 The classical Virasoro algebra for the closed bosonic string

In exercise 2.1 we showed that the components of the energy-momentum tensor in light-cone coordinates are given by

$$\begin{aligned} T_{++} &= -\frac{1}{\alpha}\partial_+ X^\mu \partial_+ X_\mu &&= 0 \\ T_{--} &= -\frac{1}{\alpha}\partial_- X^\mu \partial_- X_\mu &&= 0 \\ T_{+-} &= T_{-+} &&= 0\ . \end{aligned}$$

Remember that the components T_{++} and T_{--} vanish as a constraint and the mixed components vanish identically from the definition of the energy-momentum for the bosonic string.
Moreover, we have showed in exercise 1.3 that locally we can bring the worldsheet metric into the form $h_{\alpha\beta} = \eta_{\alpha\beta}$.

a) Show that

$$T_{++} = T_{++}(\sigma_+) \quad \text{and} \quad T_{--} = T_{--}(\sigma_-)\ .$$

(*1 Point*)

b) Show that although we have set the worldsheet metric to the form $h_{\alpha\beta} = \eta_{\alpha\beta}$ there is still a residual symmetry

$$\sigma^+ \mapsto \tilde{\sigma}^+ = \tilde{\sigma}^+(\sigma^+) \quad \text{and} \quad \sigma^- \mapsto \tilde{\sigma}^- = \tilde{\sigma}^-(\sigma^-)\ .$$

(*3 Points*)

c) Compute the associated conserved currents to the residual symmetry. (*2 Points*)

d) Determine the corresponding charges and express them through the energy-momentum tensor. (*1 Point*)

Having seen that the components of the energy-momentum tensor give rise to an infinite set of conserved charges we want to compute the algebra of these charges.

e) Use the mode expansion of the closed bosonic string from exercise 3.1 to obtain the mode expansions for the energy-momentum tensor given by

$$T_{--} = -\left(\frac{2\pi}{l}\right)^2 \sum_{n=-\infty}^{\infty} L_n \mathrm{e}^{-\frac{2\pi i n}{l}\sigma^-} \quad \text{and} \quad T_{++} = -\left(\frac{2\pi}{l}\right)^2 \sum_{n=-\infty}^{\infty} \tilde{L}_n \mathrm{e}^{-\frac{2\pi i n}{l}\sigma^+} ,$$

where we have defined the so-called *Virasoro generators*

$$L_n = \frac{1}{2}\sum_{-\infty}^{\infty} \alpha_{n-m}\cdot\alpha_m \quad \text{and} \quad \tilde{L}_n = \frac{1}{2}\sum_{-\infty}^{\infty} \tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m \ .$$

(*2 Points*)

f) Show that the Virasoro generators satisfy the centerless Virasoro algebra also known as the Witt algebra

$$\begin{aligned}
\{L_m, L_n\}_{\text{P.B.}} &= -i(m-n)L_{m+n} \\
\{\tilde{L}_m, \tilde{L}_n\}_{\text{P.B.}} &= -i(m-n)\tilde{L}_{m+n} \\
\{L_m, \tilde{L}_n\}_{\text{P.B.}} &= 0 \ .
\end{aligned}$$

(*3 Points*)

g) What does the vanishing of the energy-momentum tensor imply for the Virasoro generators? (*1 Point*)

h) Express the relativistic mass-shell condition $M^2 = -p^\mu p_\mu$ for the closed bosonic string in terms of the modes. (*1 Point*)

i) Argue that a closed string is invariant under rigid σ-translations. Give the implications of this in view of the modes. (*1 Point*)