Superstring Theory

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http://www.th.physik.uni-bonn.de/people/forste/exercises/strings19

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–Homeworks–

6.1 Lorentz symmetry of the quantum string

In exercise 1.3 you found the currents $j^{\mu\nu}$ associated to Lorentz invariance. Moreover, you analyzed the conserved charges

$$J^{\mu\nu} = \int d\sigma j^{\mu\nu}$$

in exercise 2.3 together with their mode expansion in exercise 3.1. Now in the quantum theory they become (normal-ordered) operators.
Show that

$$[L_m, J^{\mu\nu}] = 0 \ .$$

Describe the implications of this for the spectrum, in particular, in the context of representations of the Lorentz group. (*5 Points*)

6.2 and 7.1 Lorentz invariance in light-cone quantization

In the lecture you have seen the light-cone quantization where the Virasoro constraints are explicitly solved. To achieve this one makes a non-covariant gauge. Therefore, Lorentz invariance is not manifest and one hast to check that it is still a symmetry. In this exercise and the following one we will explicitly check that Lorentz invariance is present in light-cone quantization.
Throughout this exercise we will use space-time light-cone coordinates

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{d-1}) \ .$$

For the remaining transversal coordinates we use Greek letters $i = 1, \dots, d-2$. The non-vanishing components of the metric are given by

$$\eta_{-+} = \eta^{+-} = -1 \quad \text{and} \quad \eta_{ij} = \delta_{ij} \ .$$

In space-time light-cone coordinates the light-cone gauge is given by

$$X^+ = x^+ + p^+\tau \ ,$$

which means that all modes α_n^+ for $n \neq 0$ vanish. For the mode expansion of the other components we us a different convention as in exercise 3.1 which simplifies our expressions. We consider for open strings the mode expansion

$$X^\mu = x^\mu + p^\mu \tau + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu \mathrm{e}^{-in\tau} \cos n\sigma \quad \text{for } \mu \neq + \ .$$

Moreover, we set $\alpha_0^\mu = p^\mu$.

a) Show that the Virasoro constraints can be written as

$$(\dot{X} \pm X')^- = \frac{1}{2p^+} \sum_{i=1}^{d-2} (\dot{X}^i \pm X'^i)^2 \ . \tag{1}$$

(*1 Point*)

b) Solve explicitly the Virasoro constraint (1) for the modes α_n^-. You should obtain

$$\alpha_n^- = \frac{1}{p^+} \left(\frac{1}{2} \sum_{i=1}^{d-2} \sum_{m=-\infty}^{\infty} : \alpha_{n-m}^i \alpha_m^i : - a\delta_n \right) \ , \tag{2}$$

where we have introduced an unknown normal-ordering constant a; similar as for the Virasoro generator L_0. (*2 Points*)

c) Deduce the mass-shell condition from (2). (*1 Point*)

Recall the Lorentz algebra

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\mu\rho} \ J^{\nu\sigma} + \eta^{\nu\sigma} \ J^{\mu\rho} - \eta^{\nu\rho} \ J^{\mu\sigma} - \eta^{\mu\sigma} J^{\nu\rho}) \ ,$$

where the generators $J^{\mu\nu}$ can be expressed through the modes α^μ by

$$\begin{aligned} J^{\mu\nu} &= l^{\mu\nu} + E^{\mu\nu} \quad \text{with} \\ l^{\mu\nu} &= x^\mu p^\nu - x^\nu p^\mu \quad \text{and} \\ E^{\mu\nu} &= -i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu \right) \ . \end{aligned}$$

For the rest of this exercise we will focus on the commutator $[J^{i-}, J^{j-}]$ which gets a potential anomaly in light-cone gauge.

d) Show that from the Lorentz algebra $[J^{i-}, J^{j-}]$ has to vanish. (*1 Point*).

Now we will start computing the commutator $[J^{i-}, J^{j-}]$ from the mode expansion such that the constraint (2) is respected. We begin which a bunch of commutators.

e) Proof the following relation

$$[AB, CD] = A[B, C]D + AC[B, D] + [A, C]DB + C[A, D]B \ .$$

(*1 Point*)

f) Verify the following commutation relations

$$\begin{aligned}[x^-, 1/p^+] &= i(p^+)^{-2} \ , \\ [\alpha^i_m, \alpha^-_n] &= m\alpha^i_{m+n}/p^+ \quad \text{and} \\ [\alpha^-_m, x^-] &= -i\alpha^-_m/p^+ \ .\end{aligned} \tag{3}$$

(*3 Points*)

g) Argue by reference to exercise 5.2 that the commutator relation

$$[p^+\alpha^-_m, p^+\alpha^-_n] = (m-n)p^+\alpha^-_{m+n} + \left[\frac{d-2}{12}m(m^2-1) + 2am\right]\delta_{m+n} \ .$$

holds. (*1 Point*)

h) We define $E^j = p^+ E^{j-}$. Show that

$$[x^i, E^j] = -iE^{ij} \ .$$

(*2 Points*)

Begin of exercise 7.1 which has to be handed in on 29.11.2019.

i) Show that the commutator $[J^{i-}, J^{j-}]$ can be expressed as

$$\begin{aligned}[J^{i-}, J^{j-}] &= -\frac{1}{(p^+)^2}C^{ij} \quad \text{with} \\ C^{ij} &= 2ip^+p^-E^{ij} - [E^i, E^j] - iE^ip^j + iE^jp^i \ .\end{aligned} \tag{4}$$

(*2 Points*)

One can argue that the commutator $[J^{i-}, J^{j-}]$ can only contain contributions quadratic in the oscillators. More precisely, one expects the following form

$$[J^{i-}, J^{j-}] = -\frac{1}{(p^+)^2}\sum_{m=1}^{\infty}\Delta_m\left(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m\right) \ , \tag{5}$$

where the coefficitens Δ_m are complex numbers.

j) Compare equation (5) with (4) and argue that the matrix elements of C^{ij} can be used to determine the coefficients Δ_m. (*1 Point*)

We want to compute the matrix elements of C^{ij} in two steps.

k) Show that the matrix elements of C^{ij} are given by

$$\begin{aligned}
\langle 0|\alpha^k_m C^{ij}\alpha^l_{-m}|0\rangle = \langle 0| &\left(2m^2\delta^{ik}\delta^{jl} + mp^j p^k\delta^{il} - mp^j p^l\delta^{ik}\right)|0\rangle \\
&+ p^+ m\delta^{ik}\sum_{s=1}^{m}\frac{1}{s}\langle 0|\alpha^-_m\alpha^j_{-s}\alpha^l_{s-m}|0\rangle - (p^+)^2\delta^{ik}\delta^{jl}\langle 0|\alpha^-_m\alpha^-_{-m}|0\rangle \\
&+ m^2\sum_{r,s=1}^{m}\frac{1}{rs}\langle 0|\alpha^k_{m-s}\alpha^j_s\alpha^i_{-r}\alpha^l_{r-m}|0\rangle \\
&+ p^+ m\delta^{jl}\sum_{s=1}^{m}\frac{1}{s}\langle 0|\alpha^k_{m-s}\alpha^i_s\alpha^-_{-m}|0\rangle - (i\leftrightarrow j)\ .
\end{aligned} \tag{6}$$

(5 Points)

l) Compute the four matrix elements in (6). You should get the following

i) $(p^+)^2\langle 0|\alpha^-_m\alpha^-_{-m}|0\rangle = \frac{d-2}{12}m(m^2-1) + 2am$

ii) $p^+\sum_{s=1}^m \frac{1}{s}\langle 0|\alpha^-_m\alpha^j_{-s}\alpha^l_{s-m}|0\rangle = p^j p^l + \delta^{jl}m(m-1)/2$

iii) $p^+\sum_{s=1}^m \frac{1}{s}\langle 0|\alpha^k_{m-s}\alpha^i_s\alpha^-_{-m}|0\rangle = p^i p^k + \delta^{ik}m(m-1)/2$

iv) $\sum_{r,s=1}^m \frac{1}{rs}\langle 0|\alpha^k_{m-s}\alpha^j_s\alpha^i_{-r}\alpha^l_{r-m}|0\rangle - (i\leftrightarrow j) = -(m-1)(\delta^{il}\delta^{jk} - \delta^{jl}\delta^{ik})$

(4 Points)

m) Put now all together and compute the total matrix elements of C^{ij}. From this you should find

$$\Delta_m = m\left(\frac{26-d}{12}\right) + \frac{1}{m}\left(\frac{d-26}{12} + 2(1-a)\right)\ .$$

(2 Points)

From this analysis we see that in light-cone gauge the Lorentz algebra gets a potential anomaly described by Δ_m. To guarantee Lorentz invariance in light-cone gauge we have to require that the anomaly vanishes. This requires that

$$d = 26 \qquad a = 1\ ,$$

as also see from the ζ-function regularization in the lecture. Notice that we have chosen for the normal-ordering constant a the negative of the one in the lecture.