Superstring Theory

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`http://www.th.physik.uni-bonn.de/people/forste/exercises/strings19`

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–Homeworks–

7.1 Lorentz invariance in light-cone quantization

You can find the beginning of this exercise on exercise sheet 6.

i) Show that the commutator $[J^{i-}, J^{j-}]$ can be expressed as

$$\begin{aligned} [J^{i-}, J^{j-}] &= -\frac{1}{(p^+)^2} C^{ij} \quad \text{with} \\ C^{ij} &= 2ip^+p^-E^{ij} - [E^i, E^j] - iE^ip^j + iE^jp^i \ . \end{aligned} \tag{1}$$

(2 Points)

One can argue that the commutator $[J^{i-}, J^{j-}]$ can only contain contributions quadratic in the oscillators. More precisely, one expects the following form

$$[J^{i-}, J^{j-}] = -\frac{1}{(p^+)^2} \sum_{m=1}^{\infty} \Delta_m \left(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m \right) \ , \tag{2}$$

where the coefficitens Δ_m are complex numbers.

j) Compare equation (2) with (1) and argue that the matrix elements of C^{ij} can be used to determine the coefficients Δ_m. *(1 Point)*

We want to compute the matrix elements of C^{ij} in two steps.

k) Show that the matrix elements of C^{ij} are given by

$$\begin{aligned} \langle 0|\alpha^k_m C^{ij} \alpha^l_{-m}|0\rangle = \langle 0| &\left(2m^2\delta^{ik}\delta^{jl} + mp^jp^k\delta^{il} - mp^jp^l\delta^{ik} \right) |0\rangle \\ &+ p^+ m\delta^{ik} \sum_{s=1}^{m} \frac{1}{s} \langle 0|\alpha^-_m \alpha^j_{-s}\alpha^l_{s-m}|0\rangle - (p^+)^2\delta^{ik}\delta^{jl} \langle 0|\alpha^-_m\alpha^-_{-m}|0\rangle \\ &+ m^2 \sum_{r,s=1}^{m} \frac{1}{rs} \langle 0|\alpha^k_{m-s}\alpha^j_s\alpha^i_{-r}\alpha^l_{r-m}|0\rangle \\ &+ p^+ m\delta^{jl} \sum_{s=1}^{m} \frac{1}{s} \langle 0|\alpha^k_{m-s}\alpha^i_s\alpha^-_{-m}|0\rangle - (i \leftrightarrow j) \ . \end{aligned} \tag{3}$$

(5 Points)

l) Compute the four matrix elements in (3). You should get the following

   i) $(p^+)^2 \langle 0|\alpha^-_m \alpha^-_{-m}|0\rangle = \frac{d-2}{12} m(m^2-1) + 2am$

   ii) $p^+ \sum_{s=1}^{m} \frac{1}{s} \langle 0|\alpha^-_m \alpha^j_{-s} \alpha^l_{s-m}|0\rangle = p^j p^l + \delta^{jl} m(m-1)/2$

   iii) $p^+ \sum_{s=1}^{m} \frac{1}{s} \langle 0|\alpha^k_{m-s} \alpha^i_s \alpha^-_{-m}|0\rangle = p^i p^k + \delta^{ik} m(m-1)/2$

   iv) $\sum_{r,s=1}^{m} \frac{1}{rs} \langle 0|\alpha^k_{m-s} \alpha^j_s \alpha^i_{-r} \alpha^l_{r-m}|0\rangle - (i \leftrightarrow j) = -(m-1)(\delta^{il}\delta^{jk} - \delta^{jl}\delta^{ik})$

(*4 Points*)

m) Put now all together and compute the total matrix elements of C^{ij}. From this you should find

$$\Delta_m = m\left(\frac{26-d}{12}\right) + \frac{1}{m}\left(\frac{d-26}{12} + 2(1-a)\right) .$$

(*2 Points*)

From this analysis we see that in light-cone gauge the Lorentz algebra gets a potential anomaly described by Δ_m. To guarantee Lorentz invariance in light-cone gauge we have to require that the anomaly vanishes. This requires that

$$d = 26 \qquad a = 1 ,$$

as also see from the ζ-function regularization in the lecture. Notice that we have chosen for the normal-ordering constant a the negative of the one in the lecture.

7.2 The ghost system

In the lecture you considered the Faddev-Popov quantization method by fixing a gauge world-sheet metric $\hat{h}_{\alpha\beta}$. The resulting partition function reads

$$Z = \int_{\mathcal{D}} X^\mu \mathcal{D}h \, \mathrm{e}^{iS[X,h]} \longrightarrow Z = \int_{\mathcal{D}} X^\mu \mathcal{D}b\mathcal{D}c \, \mathrm{e}^{iS[X,\hat{h},b,c]} ,$$

where $b_{\alpha\beta}$ and c^α are the anti-commuting *ghost fields*. Moreover, the action is now given by

$$S[X,\hat{h},b,c] = S_P[X,\hat{h}] + S_g[\hat{h},b,c] \quad \text{with} \quad S_g[\hat{h},b,c] = -\frac{i}{2\pi}\int d^2\sigma \sqrt{-\hat{h}}\hat{h}^{\alpha\beta} b_{\beta\gamma} \hat{\nabla}_\alpha c^\gamma ,$$

where S_P is the Polyakov action and S_g is the ghost system's action. One clearly sees that it would have been inconsistent to simply set $h_{\alpha\beta} = \eta_{\alpha\beta}$ and drop the $\mathcal{D}h$ integration, as it would have neglected the ghost contribution. To appreciate the ghost contribution, one also notices that the total energy momentum tensor given by

$$T_{\alpha\beta} = T^X_{\alpha\beta} + T^g_{\alpha\beta} ,$$

now gets a contribution $T^g_{\alpha\beta}$ from the ghost action, aside of the already known contribution $T^X_{\alpha\beta}$ due to the Polykov action. Note that this modifies the central charge term in the Virasoro algebra obtained in exercise 5.2.
From now on we drop the ˆ symbol for the reference metric.

a) Compute the the energy momentum tensor $T^g_{\alpha\beta}$ of the ghost system. You should obtain

$$T^g_{\alpha\beta} = -i\left(b_{\alpha\gamma}\nabla_\beta c^\gamma + b_{\beta\gamma}\nabla_\alpha c^\gamma - c^\gamma\nabla_\gamma b_{\alpha\beta} - h_{\alpha\beta}b_{\gamma\delta}\nabla^\gamma c^\delta\right) .$$

<u>*Hint:*</u> Take into account the tracelessness of $b_{\alpha\beta}$ and the variation of ∇_α with respect to the metric $h_{\alpha\beta}$. *(2 Points)*

b) Compute the equations of motion of the ghost fields $b_{\alpha\beta}$ and c^α. *(1 Point)*

For convenience we now choose the reference metric to be the two-dimensional Minkowski metric, i.e. $\hat{h}_{\alpha\beta} = \eta_{\alpha\beta}$.

c) Write down the components of the energy momentum tensor $T^g_{\alpha\beta}$ and the equations of motion of the ghost fields $b_{\alpha\beta}$ and c^α in worldsheet light-cone coordinates. *(1 Point)*

The ghost system, consisting of Grassmann odd fields, is quantized by the following canonical anti-commutation relations

$$\{b_{++}(\sigma,\tau), c^+(\sigma',\tau)\} = 2\pi\delta(\sigma-\sigma') \quad \text{and} \quad \{b_{--}(\sigma,\tau), c^-(\sigma',\tau)\} = 2\pi\delta(\sigma-\sigma') .$$

For the closed string the solutions of the equations of motion (l-periodic in σ) are

$$c^\pm(\sigma,\tau) = \frac{l}{2\pi}\sum_{n\in\mathbb{Z}} c^\pm_n \mathrm{e}^{-\frac{2\pi}{l}in\sigma_\pm} \quad \text{and} \quad b_{\pm\pm}(\sigma,\tau) = \left(\frac{2\pi}{l}\right)^2 \sum_{n\in\mathbb{Z}} b^\pm_n \mathrm{e}^{-\frac{2\pi}{l}in\sigma_\pm} .$$

Here $b^+_n := \bar{b}_n,\ b^-_n := b_n,\ c^+_n := \bar{c}_n$ and $c^-_n := c_n$.

d) Use the anti-commutation relations to derive the anti-commutation relations of the ghost field's Fourier modes. You should get

$$\{b_m, c_n\} = \delta_{n+m} \quad \text{and} \quad \{b_m, b_n\} = \{c_m, c_n\} = 0 .$$

(1 Point)

e) Obtain the Virasoro generators of the ghost system as the conserved Noether charges[1]

$$L^g_m = -\frac{l}{4\pi^2}\int_0^l d\sigma\ T^g_{--}\mathrm{e}^{-\frac{2\pi}{l}im\sigma} \quad \text{and} \quad \bar{L}^g_m = -\frac{l}{4\pi^2}\int_0^l d\sigma\ T^g_{++}\mathrm{e}^{+\frac{2\pi}{l}im\sigma} .$$

The ghost fields $b_{\alpha\beta}$ and c^α correspond to variations perpendicular to the gauge slice and infinitesimal reparametrizations, respectively. *(1 Point)*

To fully promote Virasoro generators to quantum operators we need to take into account normal ording $:\ldots:$ of the Fourier modes. Your result above should read

$$L^g_m = \sum_{n\in\mathbb{Z}}(m-n)b_{m+n}c_{-n} \quad \longrightarrow \quad \hat{L}^g_m = \sum_{n\in\mathbb{Z}}(m-n) : b_{m+n}c_{-n} : .$$

For $m,n > 0$ the normal ordering of the ghost's Fourier modes is defined by $: b_m c_{-n} := -c_{-n}b_m$ and $: b_{-m}c_n := b_{-m}c_n$. Again we drop now the ˆ symbol for the quantum Virasoro operators L^g_m.

[1] Remember that the Noether charges are defined for $\tau = 0$.

f) Verify that $[L_m^g, L_n^g] = (m-n)L_{m+n}^g$ for $m+n \neq 0$. *(1 Point)*

Similar to exercise 5.2 normal ordering effects appear when $m+n=0$. This means the *ghost Virasoro algebra* is also a central extension of the classical ghost Virasoro algebra. A central extension $\hat{\mathfrak{g}} \coloneqq \mathfrak{g} \oplus c\mathbb{C}$ of a Lie algebra $\mathfrak{g}$ by c satisfies

- $[X,Y]_{\hat{\mathfrak{g}}} = [X,Y]_{\mathfrak{g}} + cP(X,Y)$,
- $[X,c]_{\hat{\mathfrak{g}}} = 0$ and
- $[c,c]_{\hat{\mathfrak{g}}} = 0$,

where $X, Y \in \mathfrak{g}$. This means that the element c is from the center of $\hat{\mathfrak{g}}$. Here $P : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is bilinear and anti-symmetric. Similar to the case of the Virasoro algebra of the bosonic system we note that $c^g P(L_m^g, L_n^g) = A^g(m)\delta_{m+n}$.

g) Show that $A^g(m) = -\frac{1}{12}m(26m^2 - 2)$. *(3 Points)*

Let us now look into the combined matter-ghost system. The total Virasoro generators are now given by

$$L_m = L_m^X + L_m^g + a\delta_m \ ,$$

where the last term accounts for the normal ordering ambiguity in $L_0^X + L_0^g$. Then the Virarsoro algebra of the total system follows

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n} \ .$$

h) A non-vanishing total $A(m)$ translates to an anomaly of the local Weyl transformations. Verify that this anomaly is absent if and only if $d = 26$ and $a = -1$. *(1 Point)*