Physikalisches Institut
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Theoretische Physik

Exercise Sheet 8
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Superstring Theory

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http://www.th.physik.uni-bonn.de/people/forste/exercises/strings19

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–Homeworks–

8.1 The spectrum of the quantized bosonic string

On exercise sheet 7 we determined the spacetime dimension $d = 26$ and the normal ordering constant $a = -1$ for the quantized bosonic string using two different methods.
In the following, we consider again light-cone quantization of the bosonic string. In the open bosonic theory with NN boundary conditions the mass-shell condition for a N-th excited state is given by

$$M^2 = -p_\mu p^\mu = 2p^+p^- - \sum_{i=1}^{24} p_i^2 = \frac{1}{\alpha'}(N-1) \; .$$

In closed bosonic string theory the mass-shell condition for a N-th/$\tilde{N}$-th excited state is given by

$$M^2 = -p_\mu p^\mu = 2p^+p^- - \sum_{i=1}^{24} p_i^2 = \frac{4}{\alpha'}(N-1) = \frac{4}{\alpha'}(\tilde{N}-1) \; .$$

a) Find the states for the first three levels (including the ground state) in the spectrum of open bosonic string theory with NN boundary conditions and count the number of states corresponding to their SO(24) representations for each level.
Hint: If you prefer you can also use Young tableaux from previous courses to count the number of states, i.e. the dimension of the corresponding representation. (*1 Point*)

Recall that massless states are classified by representations of SO(24), whereas massive states are classified by representations of SO(25). These are the *little groups* that appear in the quantized bosonic string theory.

b) Calculate the masses for each of the states of part a) and see into which little group their representations should fall. Count the number of states corresponding to their little gorup representations for each level and confirm that they match with the number of states you found in part a)[1]. (*2 Points*)

c) Explain what does $(L_0 - a)\,|\phi\rangle = (\tilde{L}_0 - a)\,|\phi\rangle = 0$ on a physical state $|\phi\rangle$ imply for N and $\tilde{N}$. (*1 Point*)

[1] Of course, if the little group is SO(24), the number of states matchs trivially.

d) Find the states for the first three levels (including the ground state) in the spectrum of closed bosonic string (use the level-matching condition form part c), which relates left- to right-moving modes) and count the number of states corresponding to their SO(24) representations for each level. (*1 Point*)

e) Calculate the masses for each of the states from part d) and see into which little group their representations should fall. Count the number of states corresponding to their little group representations for each level and confirm that they match with the number of states you found in part d). (*3 Points*)

8.2 The Virasoro-Shapiro amplitude

The tree-level scattering amplitude is given by the correlation function of the two-dimensional conformal field theory evaluated on the sphere

$$\mathcal{A}^{(m)} = \frac{1}{g_s^2}\frac{1}{\mathrm{Vol}}\int \mathcal{D}X\mathcal{D}h\ \mathrm{e}^{-S_P}\prod_{i=1}^{m} V_{\lambda_i}(p_i)\ ,$$

where S_P is the Polyakov action including the ghost part. Due to diffeomorphism and Weyl invariance we can fix the worldsheet metric h to be the flat metric on the plane. But there is still the residual symmetry which leaves us still with a $\mathrm{SL}(2,\mathbb{C})$ symmetry.
In this exercise we want to compute scattering amplitudes between taychons. The vertex operator for a tachyon is given by

$$V_{\text{tachyon}}(p_i) = g_s\int d^2z\ \mathrm{e}^{ip_i\cdot X}\ ,$$

where we have used complex variables for the integration, i.e. $d^2z = dzd\bar{z}$. It turns out that the ghost contribution factors out and changes only the normalization of the scattering amplitudes. Finally, we have

$$\mathcal{A}^{(m)} = \mathcal{N}\frac{g_s^{m-2}}{\mathrm{Vol}(\mathrm{SL}(2,\mathbb{C}))}\int\prod_{i=1}^{m} d^2z_i\ \langle V(z_1,p_1),\ldots,V(z_m,p_m)\rangle$$

with the expectation value $\langle\ldots\rangle$ computed using the gauged fixed Polyakov action without the ghosts

$$\langle V(z_1,p_1),\ldots,V(z_m,p_m)\rangle = \int\mathcal{D}X\ \exp\left(-\frac{1}{2\pi\alpha'}\int d^2z\ \partial X\cdot\bar{\partial}X\right)\exp\left(i\sum_{i=1}^{m}p_i\cdot X(z_i,\bar{z}_i)\right)\ .$$

a) Show that the scattering amplitude can be expressed as

$$\mathcal{A}^{(m)} = \mathcal{N}\frac{g_s^{m-2}}{\mathrm{Vol}(\mathrm{SL}(2,\mathbb{C}))}\delta^{(26)}\left(\sum_{i=1}^{m}p_i\right)\int\prod_{i=1}^{m}d^2z_i\ \exp\left(\frac{\alpha'}{2}\sum_{j,l}p_j\cdot p_l\log|z_j-z_l|\right)\ .$$

Hint: Compute the Gaussian integral using a source term J as you may know from quantum field theory. The propagator for the operator $\partial\bar{\partial}$ satisfies the relation

$$\partial\bar{\partial}G(z,\bar{z};w,\bar{w}) = \delta(z-w)\delta(\bar{z}-\bar{w}) \tag{1}$$

and is given in two dimensions by

$$G(z,\bar{z};w,\bar{w}) = \frac{1}{2\pi}\log|z-w|^2 \ .$$

Take special care about the zero modes. (*2 Points*)

There is a problem with equation (1) if $j = l$. Actually, this case has to be left out which can be shown by a more careful analysis implementing normal ordering for the vertex operators. One then finds our final form of the amplitude

$$\mathcal{A}^{(m)} = \mathcal{N}\frac{g_s^{m-2}}{\mathrm{Vol}(\mathrm{SL}(2,\mathbb{C})}\delta^{(26)}\left(\sum_{i=1}^{m} p_i\right)\int\prod_{i=1}^{m} d^2 z_i \prod_{j<l}|z_j - z_l|^{\alpha' p_j\cdot p_l} \ .$$

In the following we focus on $2 \to 2$ scattering of tachyons. We can fix the residual $\mathrm{SL}(2,\mathbb{C})$ by requiring

$$z_1 = \infty \ , \quad z_2 = 0 \ , \quad z_3 = z \quad \text{and} \quad z_4 = 1 \ .$$

b) Show that then

$$\mathcal{A}^{(4)} = \mathcal{N}g_s^2\delta^{(26)}\left(\sum_{i=1}^{4} p_i\right)\int d^2 z \ |z|^{\alpha' p_2\cdot p_3} \ |1-z|^{\alpha' p_3\cdot p_4} \ .$$

(*1 Point*)

c) Use Mandelstam variables $s = -(p_1+p_2)^2$, $t = -(p_1+p_2)^2$ and $u = -(p_1+p_4)^2$ to show that

$$\mathcal{A}^{(4)} = \mathcal{N}g_s\delta^{(26)}\left(\sum_{i=1}^{4} p_i\right)\frac{\Gamma(-1-\frac{\alpha' s}{4})\Gamma(-1-\frac{\alpha' t}{4})\Gamma(-1-\frac{\alpha' u}{4})}{\Gamma(2+\frac{\alpha' s}{4})\Gamma(2+\frac{\alpha' t}{4})\Gamma(2+\frac{\alpha' u}{4})} \ , \tag{2}$$

which is also called the *Virasoro-Shapiro amplitude.*
<u>*Hint:*</u> First show that $s+t+u = \sum_{i=1}^{4} = M_i^2 = -\frac{16}{\alpha'}$. Furthermore, the following integral might be useful

$$\int d^2 z \ |z|^{2a-2} \ |1-z|^{2b-2} = 2\pi\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)} \ . \tag{3}$$

(*3 Points*)

One immediately sees that the *Virasoro-Shapiro amplitude* (2) is totally symmetric in alle three Mandelstam variables s, t and u.

d) Fix t and u such that only s varies. Determine the locations of the poles in s. Give a physical interpretation of the pole locations. (*1 Point*)

e) Now we want to consider the kinematic limit of high energies where the angle θ between incoming and outgoing particles is kept fixed, $s \to \infty$ and $t \to \infty$ but the ratio $\frac{s}{t}$ is fixed. Show that

$$\mathcal{A}^{(4)} \sim \exp\left(-\frac{\alpha'}{2}(s\log s + t\log t + u\log u)\right) \quad \text{as } s \to \infty$$

and being away from the poles. Interpret this result. (*1 Point*)

f) Proof the identity (3).
Hint: Start with the identity (proof this!)

$$|z|^{2a-2} = \frac{1}{\Gamma(1-a)} \int_o^\infty dt\ t^{-a} \mathrm{e}^{-|z|^2 t}.$$

Then evaluate the arising Gaussian integrals. Afterwards it might be useful to make a coordinate transformation of the form $t = \alpha\beta$ and $u = (1-\beta)\alpha$. Determine the ranges of α and β. *(3 Points)*

Useful formulas:

$$\Gamma(z) = \int_0^\infty dt\ t^{z-1} \mathrm{e}^{-t}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}\ , \quad \text{for } z \in \mathbb{C} \backslash \mathbb{Z}$$

$$\Gamma(z) \sim \exp(z \log z)\ , \quad \text{for } z \to \infty$$

$$B(x,y) = \int_0^1 dt\ t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$