Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise Sheet 9
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Superstring Theory

Priv.-Doz. Dr. Stefan Förste und Christoph Nega

http://www.th.physik.uni-bonn.de/people/forste/exercises/strings19

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–HOMEWORKS–

9.1 State degeneracy

Consider light-cone gauge quantization of open bosonic string theory with NN boundary conditions.

a) Show that states $|\phi\rangle$ corresponding to $\hat{N}\,|\phi\rangle = N\,|\phi\rangle$, where $\hat{N} = \sum_{i=1}^{24}\sum_{n=1}^{\infty}\hat{\alpha}_{-n}^{i}\hat{\alpha}_{n}^{i}$ is the number operator and $N = \sum_{l=1}^{k} n_l$, have the form

$$|\phi\rangle = \hat{\alpha}_{-n_1}^{i_1}\cdots\hat{\alpha}_{-n_k}^{i_k}\,|0,p^\mu\rangle \ ,$$

where $|0,p^\mu\rangle$ is the ground state with momentum p^μ. (*1 Point*)

b) Consider the one-dimensional case, for example $i = 1$ in $\hat{N}$ of item a) without loss of generality. Show that the number of states, i.e. partitions or degeneracy, at a N-th level of a single family of oscillators is given by the coefficient of q^N in

$$\sum_{N=0}^{\infty} P(N) q^N = \prod_{i=1}^{\infty}(1-q^n)^{-1} \ . \tag{1}$$

(*2 Points*)

Expression (1) will appear several times when discussing partition functions in string perturbation theory. Moreover, it is related with the Dedekind η-function, defined by

$$\eta(\tau) = q^{1/24}\prod_{i=1}^{\infty}(1-q^n) \quad \text{with } q = \mathrm{e}^{2\pi i\tau} \ ,$$

where τ is the modulus of a torus T^2.

9.2 $\mathrm{SL}(2,\mathbb{Z})$ transformations and the moduli space of a torus

In string perturbation theory the relevant Riemann surface for computing the one-loop partition function of the closed string is given by a two-dimensional torus T^2 (this is a vacuum diagram since there are no external strings). Let us therefore study how to identify two conformally inequivalent tori.

T^2 is defined by modding out the complex z-plane $\mathbb{C}$ by a two-dimensional lattice $\Lambda = \{n +$

$m\tau | n, m \in \mathbb{Z}\}$ with generating lattice vectors 1 and τ such that $z \sim z + n + m\tau$, i.e. $T^2 = \mathbb{C}/\Lambda$. $\tau \in \mathbb{C}$ is the torus *modulus* or *Teichmüller parameter* describing points in the *Teichmüller space* (for a torus T^2 it is the upper-half plane $\mathbb{H}_+ = \{\tau \in \mathbb{C} | \mathrm{Im}(\tau) > 0\}$).
The torus modulus τ changes under global diffeomorphisms called *modular transformations* $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$ but the torus is left invariant. Therefore, the *moduli space* of the torus (:= the space of conformally inequivalent tori) is given by

$$\mathcal{M}_{T^2} = \frac{\text{Teichmüller space}}{\text{Modular group}} \ .$$

The group $\mathrm{SL}(2,\mathbb{Z})$ is represented by 2×2 matrices with unit determinant and integer elements

$$\mathrm{SL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\} \ .$$

It acts on $z \in \mathbb{C}$ by Möbius transformations $z \mapsto z' = \frac{az+b}{cz+d}$. The generators of the group $\mathrm{SL}(2,\mathbb{Z})$ are given by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ .$$

a) Describe the action of T and S on the upper-half plane $\mathbb{H}_+$. Moreover, give their geometrical interpretations. (*2 Points*)

b) Argue that it suffice to restrict to $\mathrm{PSL}(2,\mathbb{Z})$ instead of $\mathrm{SL}(2,\mathbb{Z})$. (*1 Point*)

c) Show that a choice for the *fundamental domain* $\mathcal{F}$ of the torus, i.e. a subset of the upper-half plane $\mathbb{H}_+$ such that any point in $\mathbb{H}_+$ is related to a point in $\mathcal{F}$ by the action of $\mathrm{PSL}(2,\mathbb{Z})$, is given by

$$\mathcal{F} = \left\{ z \in \mathbb{H}_+ | \ -\frac{1}{2} \leq \mathrm{Re}(\tau) \leq 0 \ , \ |\tau| \geq 1 \right\} \cup \left\{ z \in \mathbb{H}_+ | \ 0 < \mathrm{Re}(\tau) < \frac{1}{2} \ , \ |\tau| > 1 \right\} \ .$$

<u>*Hint:*</u> Start with τ possibly outside of the fundamental domain $\mathcal{F}$ and act k times with T to get $\mathrm{Re}(\tau)$ in a strip of width 1. Then use S conveniently. (*3 Points*)

d) Calculate how $d^2\tau$ and $\mathrm{Im}(\tau)$ transform under Möbius transformations of $\mathrm{PSL}(2,\mathbb{Z})$. Find an expression (not the constant function) in $d^2\tau$ and $\mathrm{Im}(\tau)$ which is invariant under $\mathrm{PSL}(2,\mathbb{Z})$ transformations. (*3 Points*)

e) Show that the Dedekind η-function defined in exercise 9.1 transforms under T by

$$\eta(\tau + 1) = \mathrm{e}^{i\pi/12} \ \eta(\tau) \ .$$

(*2 Points*)

<u>*Bonus exercise:*</u>

f) Show that the Dedekind η-function transforms under S by

$$\eta(-1/\tau) = \sqrt{-i\tau} \ \eta(\tau) \ ,$$

which together with part e) gives the transformation behaviour of the Dedekind η-function under the whole modular group $\mathrm{PSL}(2,\mathbb{Z})$.

Hint: See also Neal Koblitz "Introduction to Ellipitc Curves and Modular Forms". You can use the following property of the second Eisenstein series

$$E_2(-1/\tau) = z^2 E_2(\tau) + \frac{12}{2\pi i} z \ .$$

(*+4 Extra points*)