Physikalisches Institut
Universität Bonn
Theoretische Physik

Übungsblatt 1
12. Oktober 2012
WS 12/13

Übungen zu Theoretische Physik IV

Priv.-Doz. Dr. Stefan Förste

http://www.th.physik.uni-bonn.de/people/forste/exercises/ws1213/tp4

–In-class exercises–

A 1.1 Reminder: Probability Theory

We start with a little reminder about probability theory, in which we are going to discuss the basic notions and concepts that are important in the context of statistical physics.

(a) The *sample space* E of an experiment is the space of all possible outcomes of that experiment. Then a *random variable* is defined as a map $X : E \to \Omega$, with a set Ω. Consider the experiment of throwing a dice twice. Define a valid random variable for this experiment.

(b) The *probability distribution* P_X of the random variable X is a map $P_X : \Omega_X \to [0,1] \subset \mathbb{R}$, with the normalization property

$$\sum_{e \in E} P_X(x(e)) = 1$$

and which displays the probability of the random variable X to take a specific value $x(e)$. Calculate the probability distribution of the random variable you defined in (a) and check its normalization.

(c) When repeating an experiment infinitely many times, the average value of the random variable X is given by its *expectation value*

$$\langle X \rangle \equiv \sum_{e \in E} x(e) \cdot P_X(x(e)).$$

Calculate the expectation value of the random variable you defined in (a).

(d) The *variance* of a random variable is defined as

$$(\Delta X)^2 \equiv \langle (X - \langle X \rangle)^2 \rangle$$

and denotes the expectation value of the squared deviation of that random variable from its expected value. Show, that

$$(\Delta X)^2 = \langle X^2 \rangle - \langle X \rangle^2$$

holds.

(e) In order to describe the (in)dependence of two given random variables X_i and X_j, one defines the *correlation coefficient*

$$K_{ij} \equiv \langle (X_i - \langle X_i \rangle)\, (X_j - \langle X_j \rangle) \rangle \,.$$

The two random variables are denoted independent (uncorrelated) if their correlation coefficient vanishes. Consider the following (stochastic) experiments

- A dice is thrown twice. The sample space is given by $E = \{(1,1), (1,2), \ldots, (6,6)\}$ and we define two random variables $X_1^1 : (a,b) \mapsto a$ and $X_2^1 : (a,b) \mapsto b$.
- A physicist shoots at a soccer practice target twice. The probability of him scoring is given by 50% for the first shot. If he scores the probability of him scoring a second time is again given by 50%. In he doesn't score in the first shot, he becomes nervous and the probability of him scoring a in his second try drops to 25%. Again we define two random variables: X_1^2 is 1 in case of the physicist scoring in his first try and 0 else; X_2^2 is 1 in case of the physicist scoring in his second try and 0 else.

Calculate the correlation coefficient of X_1^1, X_2^1 as well as the one of X_1^2, X_2^2.

(f) Up to now we have looked at experiments which were performed a single time. When performing an experiment several times one can ask for the probability of a random variable X to take the values x_i (with probabilities $P_X(x_i) \equiv p_i$) k_i times (here $i = 1, \ldots, n$ and $\sum_i k_i = N$). This probability is given by the *multinomial distribution*

$$P(\{p_i\}, \{k_i\}) = \begin{cases} \frac{N!}{k_1!k_2!\ldots k_n!}\, p_1^{k_1}\, p_2^{k_2} \ldots p_n^{k_n}, & \text{if} \quad \sum_i k_i = N, \\ 0 & \text{else.} \end{cases}$$

For *Bernoulli processes*, that means experiments with exactly two possible outcomes, the multinomial distribution becomes the *binomial distribution*

$$B(p,k) = \binom{N}{k} p^k (1-p)^{N-k}.$$

$B(p,k)$ gives the probability of the random variable X to take the value x in k of N cases and p is the probability of X taking the value x when performing the experiment once. Calculate the expected value and the variance of the random variable k with probability distribution $B(p,k)$.
Hint: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

A 1.2 Reminder: Quantenmechanik
This exercise is intended to be a little reminder about quantum mechanics. We want to remind you of the basic notions and concepts which are going to be needed in the context of quantum statistics.
In quantum mechanics the classical observables such as position x and momentum p are replaced by hermitean operators which act on states of the *Hilbert space*. If a system is in the state $|\Psi\rangle$, the *expectation value* of the observable O when performing a measurement

is given by $\langle\Psi|O|\Psi\rangle$. By projecting the state $|\Psi\rangle$ onto eigenvectors of an operator O one can define the (complex) wave function $\Psi(o) = \langle O|\Psi\rangle$, whose absolute square gives the *probability density* $P(o) = |\Psi(o)|^2$ for the measurement of the observable O. Normalization of the probability density is then a direct consequence of the normalization of the states in the Hilbert space. The time-evolution of a state is determined by the *Schrödinger equation* $\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi\rangle = H|\Psi\rangle$, with the Hamilton operator H being the operator used to measure the energy of a given state.

(a) Show, that the expectation value of the position operator is given by $\langle X\rangle = \int x\Psi^*(x)\Psi(x)\mathrm{d}x$. This is the continuous version of the expectation value as we defined it in exercise A 1.1.

(b) When measuring the observable O, the quantum mechanical state $|\Psi\rangle$ is projected onto an eigenstate of the operator O (wave function collapse). As a consequence it is for instance impossible to determine position and momentum of a particle with unlimited precision at the same time. In case two operators commute, it is possible, though, to find a common eigenbasis for these two operators. In the context of quantum mechanics this means, that measurements of the corresponding two observables do not influence each other.
Show, that for two non-commuting (hermitean) operators A and B, the following uncertainty relation holds:

$$\frac{1}{2}\left(\langle AB\rangle - \langle BA\rangle\right) \leq \Delta A \cdot \Delta B.$$

Here ΔA is the square root of the variance of the observable (random variable) A.

Consider now, as an example for a quantum mechanical system, the harmonic oscillator. The Hamilton operator is given by

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2,$$

where m is the mass of the oscillator and ω describes the frequency of the oscillation.

(c) In this context it is useful to define two new operators a and $a^\dagger$ as

$$a \equiv \left(\frac{m\omega}{2\hbar}\right)^{1/2} X + \mathrm{i}\left(\frac{1}{2m\omega\hbar}\right)^{1/2} P,$$
$$a^\dagger \equiv \left(\frac{m\omega}{2\hbar}\right)^{1/2} X - \mathrm{i}\left(\frac{1}{2m\omega\hbar}\right)^{1/2} P.$$

Show, starting from the canonical commutation relation $[X, P] = i\hbar$, that a and $a^\dagger$ fulfill $\left[a, a^\dagger\right] = 1$ and express the Hamilton operator in terms of these operators.

(d) Now, define the number operator $N \equiv a^\dagger a$ and the corresponding (normalized) eigenstates $N|n\rangle = n|n\rangle$. Show, that

$$a|n\rangle = \sqrt{n}\,|n-1\rangle\,,$$
$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle\,.$$

(e) Consider a state[1] $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|n\rangle + |n+1\rangle\right)$. Show, that

$$\langle\Psi|X(t)|\Psi\rangle \propto \cos(ft),$$

with a constant f. Hence, in dynamical systems the expectation value of an obervable, or random variable, can be time-dependent.

(f) Calculate the expectation value $\langle X^2(t)\rangle$ as well as the variance $(\Delta X(t))^2$.

–Homework–

H 1.1 Spin system *5 Points*

Consider a system of N non-interacting particles each of which is in the state $|\uparrow\rangle$ or $|\downarrow\rangle$ with equal probability.

(a) What is the probability to find exactly N particles in the state $|\uparrow\rangle$?

(b) Let the magnetization M be defined as $M \equiv 2k - N$. Calculate its expectation value as well as its variance.

H 1.2 Binomial- and Poisson distribution *5 Points*

Show, that in the limit $N \to \infty$, $p \to 0$, keeping $\mu = Np$ constant, the binomial distribution $B(k,p)$ approaches the *Poisson distribution*:

$$\lim_{\substack{N\to\infty \\ p\to 0 \\ Np=\text{const.}}} B(k,p) = \frac{\mu^k}{k!}\mathrm{e}^{-\mu}.$$

Hint: $\lim_{p\to 0}(1-p)^{-\frac{1}{p}} = \mathrm{e}$

H 1.3 Drunk Physicists *10 Punkte*

After the physicist's party[2] in the winter term 2012/2013, two drunk students leave the Carpe Noctem. Both of them start at the exit and for both of them the probabilities to make a step to the left or one to the right are equal. They always make their (equally sized) steps at the same time.

(a) What is the probability that after N steps, they have made k steps away from and l steps towards each other?

(b) What is the probability for them to meet again after N steps?

[1] In the Heisenberg picture, the states are time-independent. Then the operators carry the time-dependence: If the Hamilton operator is itself time-independet, the time-dependence of an operator $O(t)$ is determined by $O(t) = \mathrm{e}^{-\mathrm{i}Ht/\hbar}\, O\, \mathrm{e}^{\mathrm{i}Ht/\hbar}$.

[2] on the 17th of october 2012 from 10pm on