Physikalisches Institut
Universität Bonn
Theoretische Physik

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Übungen zu Theoretische Physik IV

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http://www.th.physik.uni-bonn.de/people/forste/exercises/ws1213/tp4

–In-class exercises–

A 2.1 Saddle-point method

In the context of statistical physics it is often necessary to solve integrals of the form

$$I = \lim_{N\to\infty} \int_a^b \mathrm{e}^{Nf(x)} \mathrm{d}x.$$

If $f(x)$ is an analytic function on the interval $[a,b]$ and has a global minimum at $x_0 \in (a,b)$ then

$$I = \lim_{N\to\infty} \mathrm{e}^{Nf(x_0)} \sqrt{\frac{2\pi}{N|f''(x_0)|}}, \qquad \text{where} \quad f''(x_0) \equiv \left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_0}.$$

Show *Stirling's formula*

$$N! \to \sqrt{2\pi N} N^N \mathrm{e}^{-N} \qquad \text{as} \quad N \to \infty$$

using the saddle-point method.
Hint: Use the integral representation of the gamma function $N! = \Gamma(N+1) = \int_0^\infty x^N \mathrm{e}^{-x} \mathrm{d}x$.

A 2.2 Ensemble of quantum mechanical harmonic oszillators

Consider a system of N distinguishable, non-interacting, quantum mechanical, harmonic oscillators with equal angular velocity ω. States of the complete system are then given by the individual oscillator states

$$|n_1, n_2, \ldots, n_N\rangle = |n_1\rangle \otimes |n_2\rangle \otimes \cdots \otimes |n_N\rangle .$$

We write shorthand

$$a_i \equiv \mathbb{1}^{\otimes(i-1)} \otimes a \otimes \mathbb{1}^{\otimes(N-i)} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \underset{\substack{\uparrow \\ i\text{te Stelle}}}{a} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

for the *lowering operator* of the ith oscillator (with similar expressions for $a_j^\dagger$, N_j, H_j). The Hamilton operator of the system is given by

$$H = \sum_{j=1}^{N} \hbar\omega \left(a_j^\dagger a_j + \frac{1}{2} \right).$$

First, consider the case $N = 3$ with total energy $E = \frac{9}{2}\hbar\omega$ of the system.

(a) How many different states realize this value of the energy?

(b) What is the probability $p(\epsilon)$ for a given oscillator to have the energy ϵ?

Now we want to determine the number of states for a given energy E in the limit of a very big number of oscillators $N \gg 1$. In general, it is given by

$$\Omega(E) \equiv \mathrm{Sp}\, \delta(E - H).$$

(c) What is $\Omega(E)$ for the given system?

(d) Show, that

$$\Omega(E) = \int \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{\mathrm{i}kE} \left(\frac{\mathrm{e}^{-\mathrm{i}k\hbar\omega/2}}{1 - \mathrm{e}^{-\mathrm{i}k\hbar\omega}} \right)^N$$

and further, that

$$\Omega(E) = \int \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{N(\mathrm{i}k(E/N) - \log(2\mathrm{i}\sin(k\hbar\omega/2)))}.$$

(e) This integral can be computed using the saddle-point method. Show, that $\Omega(E)$ is given by

$$\Omega(E) = \exp\left\{ N \left[\frac{\frac{E}{N} + \frac{1}{2}\hbar\omega}{\hbar\omega} \log \frac{\frac{E}{N} + \frac{1}{2}\hbar\omega}{\hbar\omega} - \frac{\frac{E}{N} - \frac{1}{2}\hbar\omega}{\hbar\omega} \log \frac{\frac{E}{N} - \frac{1}{2}\hbar\omega}{\hbar\omega} \right] \right\}.$$

–Homework–

H 2.1 Spin precession of a spin-1/2 particle *(5+5=10) Points*

The Hamilton operator of a spin-1/2 particle in a homogeneous magnetic field B is given by

$$H = -\frac{\gamma}{2}\hbar \sum_{j=1}^{3} \sigma_j B_j,$$

where γ is the gyromagnetic constant and σ_i $(i = 1, 2, 3)$ are the Pauli matrices.
The time variation of the polarisation is given by

$$\frac{\partial P_i}{\partial t} = \frac{\partial \langle \sigma_i \rangle}{\partial t}.$$

Here, σ_i is to be understood as an operator in the spin-1/2 representation of the rotation group.

(a) Express the time variation of the polarisation in terms of the density matrix $\rho(t)$ and show

$$\mathrm{i}\frac{\partial P_i}{\partial t} = -\frac{\gamma}{2}\sum_j B_j \,\mathrm{Sp}\left([\sigma_i, \sigma_j]\rho\right).$$

(b) Show the *Bloch equation*

$$\frac{\partial}{\partial t}P = \gamma\left(P \times B\right).$$

Hint: $[\sigma_i, \sigma_j] = 2i\sum_k \sigma_k \epsilon_{ijk}$.

H 2.2 Spin ensemble *(4+2+2+2=10) Points*

Consider a system of N ($N \gg 1$) non-interacting spin-1/2 particles in a constant magnetic field B. Each of the particles has a magnetic moment μ, which can be aligned parallelly or anti-parallelly to the magnetic field. Let n_1 (n_2) be the number of magnetic moments aligned parallelly (anti-parralelly) to the magnetic field. The energy of the system is then given by $E = -(n_1 - n_2)\mu B$.

(a) Show, that the number of states having an energy between E and $E + \delta E$ is given approximately by

$$\omega(E, \delta E) = \frac{N!}{\left(\frac{N}{2} - \frac{E}{2\mu B}\right)!\left(\frac{N}{2} + \frac{E}{2\mu B}\right)!}\frac{\delta E}{2\mu B},$$

where $E \gg \delta E \gg \mu B$.
Hint: What is the energy difference between two energy levels?

(b) Use Stirling's formula, as it was derived in exercise A 2.1, to find an approximation to $\ln\omega(E, \delta E)$.

(c) Interprete the function $\ln f(n_1) \equiv \ln\left(\frac{N!}{n_1!(N-n_1)!}\right)$ as a continuous function of n_1. Its Taylor expansion around the maximum $n_{1,\max}$ up to second order is then given by

$$\ln f(n_1) = \ln f(n_{1,\max}) + \frac{1}{2}(n_1 - n_{1,\max})^2 \left.\frac{\partial^2 \ln f(n_1)}{\partial n_1^2}\right|_{n_1 = n_{1,\max}}.$$

Using the approximation $\ln n! \approx n\ln n - n$, calculate the maximum $n_{1,\max}$ as well as $\left.\frac{\partial^2 \ln f(n_1)}{\partial n_1^2}\right|_{n_1 = n_{1,\max}}$.

(d) Use the exponential of the Taylor expansion of $\ln f(n_1)$ to show that $\omega(E, \delta E)$ is given approximately by the Gaussian distribution

$$\omega(E, \delta E) = 2^N\sqrt{\frac{2}{\pi N}}\frac{\delta E}{2\mu B}\exp\left[-\frac{2}{N}\left(\frac{E}{2\mu B}\right)^2\right].$$