Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 4
11. November 2013
WS 13/14

Exercises on Group Theory

Priv.-Doz. Dr. Stefan Förste

–Home Exercises–

H 4.1 $SO(3)$ Representation Product *(15 points)*

The fundamental, defining and three-dimensional irreducible representation D_{F} of $SO(3)$ acts on a vector $\phi \in \mathbb{R}^3$ as

$$\phi^i \mapsto (D_{\mathrm{F}}(R)\phi)^i = R^i_{\ j}\phi^j\,, \qquad R \in SO(3)\,.$$

We take the tensor product of this representation with itself. It acts on

$$\Phi^{ij} \in \mathbb{R}^3 \otimes \mathbb{R}^3 \cong \mathbb{R}^9$$

as

$$\Phi^{ij} \mapsto (D_{\mathrm{F}\otimes\mathrm{F}}(R)\Phi)^{ij} = \mathscr{R}^{ij}_{\ \ kl}\Phi^{kl}$$

(a) Express the matrix components $\mathscr{R}^{ij}_{\ \ kl}$ of $D_{\mathrm{F}\otimes\mathrm{F}}$ in terms of $R^i_{\ j}$. *(1 point)*

(b) Consider the following operators acting on $\mathbb{R}^9$:

$$\begin{aligned}
(\mathscr{P}_0)^{ij}_{\ \ kl} &= \frac{1}{3}\delta^{ij}\delta_{kl}\,,\\
(\mathscr{P}_1)^{ij}_{\ \ kl} &= \frac{1}{2}\left(\delta^i_k\delta^j_l - \delta^i_l\delta^j_k\right)\,,\\
(\mathscr{P}_2)^{ij}_{\ \ kl} &= \frac{1}{2}\left(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k\right) - \frac{1}{3}\delta^{ij}\delta_{kl}\,.
\end{aligned}$$

Show that they form a complete set of projection operators on $\mathbb{R}^9$, i.e. that $\mathscr{P}_i\mathscr{P}_j = \delta_{ij}\mathscr{P}_i$ and $\sum_i \mathscr{P}_i = \mathbb{1}$. *(5 points)*

(c) Show that $[\mathscr{P}_i, \mathscr{R}] = 0$ for $i = 0, 1, 2$. *(3 points)*

(d) What does $\mathscr{P}_i \not\propto \mathbb{1}$ then imply using Schur's lemma? What information do you get about the spaces projected on? *(1.5 points)*

(e) How do the matrices $\mathscr{P}_i\Phi$ look like? What is the dimension of $\mathscr{P}_i\mathbb{R}^9$? *(4.5 points)*

At the end you should recover the decomposition

$$|l=1\rangle \otimes |l=1\rangle = |l=0\rangle \oplus |l=1\rangle \oplus |l=2\rangle \tag{1}$$

which you should know from angular momentum addition in quantum mechanics.

H 4.2 Representation of S_3, part 2 *(6 points)*

Remember last sheet where you constructed a matrix representation of S_3 acting on $\langle v_1 + v_2 + v_3\rangle^\perp$.

(a) Show the matrix representation you found in H3.2a) is unitary. (*1 point*)

(b) Using your basis $\{e_1, e_2\}$ of $\langle v_1 + v_2 + v_3\rangle^\perp$, compute the matrix $A_{ij} = \langle e_i, e_j\rangle_{\mathbb{R}^3}$ where $\langle \cdot, \cdot\rangle_{\mathbb{R}^3}$ denotes the scalar product in $\mathbb{R}^3$ which makes $\{v_i\}$ an orthonormal basis. (*1 point*)

(c) Show that using A as a scalar product, the representation of S_3 acting on $\langle v_1+v_2+v_3\rangle^\perp$, $\hat{D}$, is a unitary representation, i.e. that

$$\hat{D}(\sigma)A\hat{D}(\sigma)^T = A\,, \qquad \text{for all } \sigma \in S_3\,.$$

(*4 points*)

H 4.3 Intertwiner *(4 points)*

Show that the space of all self-intertwiners of the fundamental representation of $SO(2)$ is isomorphic to $\mathbb{C}$.