Physikalisches Institut
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Theoretische Physik

Exercise 6
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Exercises on Group Theory

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–Home Exercises–

Let G be a finite, discrete group, V_μ and V_ν vector spaces of dimensions d_μ and d_ν respectively. Let $D_{(\mu)}$ and $D_{(\nu)}$ be two irreducible representations of G, such that

$$D_{(\mu)}(g)\ :\ \begin{aligned} V_\mu &\longrightarrow V_\mu \\ v^i &\longmapsto \sum_k \left(D_{(\mu)}(g)\right)^i{}_k v^k\,, \end{aligned}$$

and analogously for $D_{(\nu)}(g)$. We will see in the lecture that the following *orthogonality theorem* holds,

$$\sum_{g\in G} \left(D_{(\mu)}(g)\right)^i{}_k \left(D_{(\nu)}(g^{-1})\right)^l{}_j = \frac{|G|}{d_\mu}\delta_{\mu\nu}\delta^i_j\delta^l_k\,.$$

For a given representation $D_{(\mu)}$, the *character* of this representation is defined as

$$\chi_{(\mu)}\ :\ \begin{aligned} G &\longrightarrow \mathbb{C} \\ g &\longmapsto \chi_{(\mu)}(g) = \operatorname{tr} D_{(\mu)}(g) \end{aligned}\,.$$

Since the characters of all group elements within one conjugacy class are clearly equal we can denote the character of a conjugacy class $[g]$ by $\chi_{(\mu)}([g]) = \chi_{(\mu)}(g)$.
A *character table* of a given group is then a table of the form

	C_1	$\dots$	C_n
$D_{(1)}$	$\chi_{(1)}(C_1)$	$\dots$	$\chi_{(1)}(C_n)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$D_{(n)}$	$\chi_{(n)}(C_1)$	$\dots$	$\chi_{(n)}(C_n)$

,

where $C_1, \dots C_n$ denote the conjugacy classes of that group. We will show in the lecture, that the number of irreducible representations always equals the number of conjugacy classes of a given (finite) group, so that character tables always have the same number of columns and rows.

H 6.1 $\mathbb{Z}_N$ irreps *(6 points)*

(a) Find all irreducible representations of the cyclic group $\mathbb{Z}_N$.

Hints: What does the Abelianity of $\mathbb{Z}_N$ imply about the dimensions of the representation spaces? What does finiteness of $\mathbb{Z}_N$ imply? Use the formula

$$N = \sum_\mu n_\mu^2$$

where the sum runs over all irreducible representations μ and n_μ is the dimension. *(4 points)*

(b) Use the orthogonality theorem to deduce the formula

$$\frac{1}{N}\sum_{j=0}^{N-1} e^{2\pi i n j/N} e^{-2\pi i n' j/N} = \delta_{nn'}$$

(2 points)

H 6.2 Inverse Conjugacy classes *(2 points)*

Let G be any group. Show that for each conjugacy class there is one conjugacy class containing all the inverses.

H 6.3 Group algebra *(3 points)*

Let K be a field and let A be a vector space over K, equipped with an additional (binary) operation $* : A \times A \to A$. Then A is an associative algebra over K iff $*$ is bilinear and associative, i.e. the following identities hold $\forall x, y, z \in A, a, b \in K$:

(i) $(x + y) * z = x * z + y * z$ (*left-distributivity*),

(ii) $z * (x + y) = z * x + z * y$ (*right-distributivity*),

(iii) $(ax) * (by) = (ab)(x * y)$,

(iv) $(x * y) * z = x * (y * z)$ (*associativity*).

Let G be a group. Then we can define the $\mathbb{C}$ vector space V_G, the vectors of which are given by

$$V_G \ni v = \sum_{g\in G} v_g g\,, \qquad \text{where} \qquad \forall g \in G\ v_g \in \mathbb{C}\,.$$

The group elements g naturally form a basis of this vector space. Define the additional operation

$$* : \quad \begin{aligned} V_G \times V_G &\longrightarrow V_G \\ (v, w) &\longmapsto \sum_{g,h\in G} v_g w_h\, g \cdot h\,, \end{aligned}$$

where $\cdot$ denotes the group product.
Show that V_G is a vector space and that together with $*$ it forms an associative algebra.

H 6.4 Characters *(8 points)*

(a) Let G be a finite, discrete group. Let $D_{(\mu)}$ and $D_{(\nu)}$ be two irreducible representations. Show the *orthogonormality theorem for characters*,

$$\sum_{g\in G} \chi_{(\mu)}(g)\chi^*_{(\nu)}(g) = |G|\delta_{\mu\nu}\,.$$

(*4 points*)

(b) Show that S_3 has three irreducible representations. What are their dimensions? (*1 point*)

(c) Compute the character table of S_3. *Hint: Ex. H3.2(f)* (*3 points*)